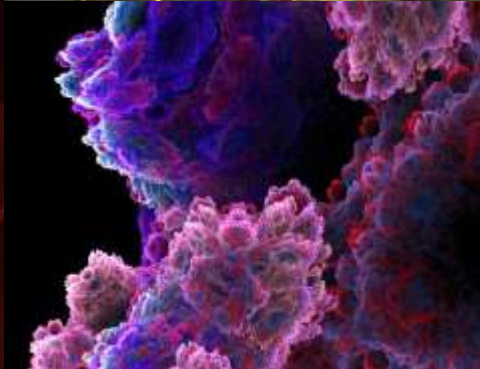
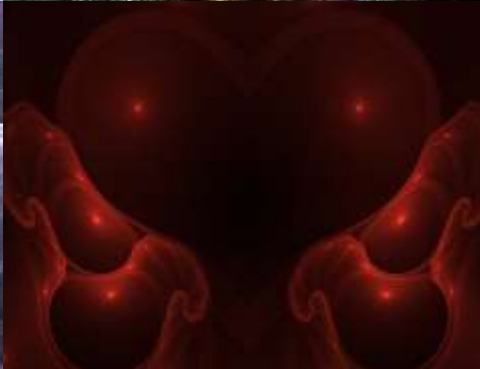
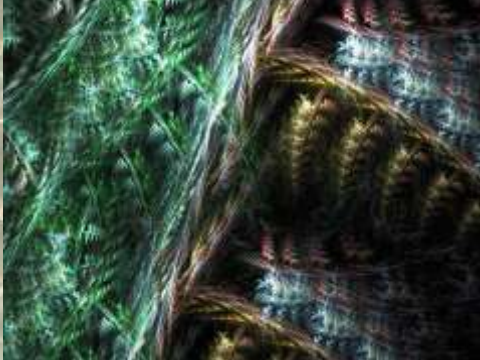


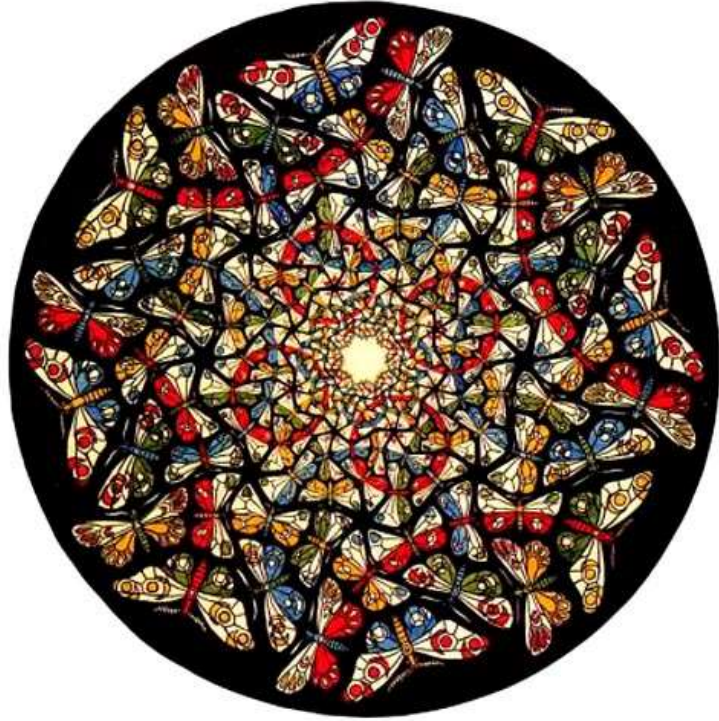
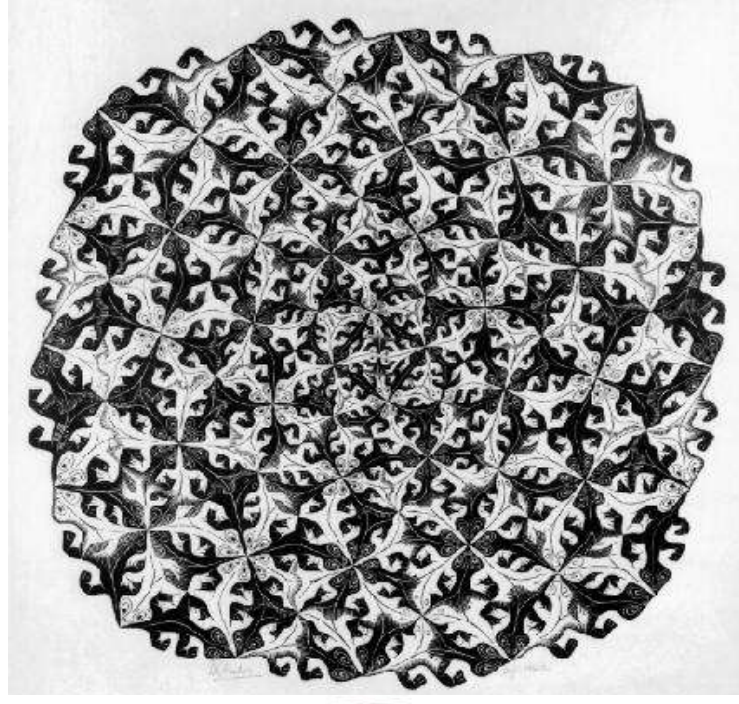
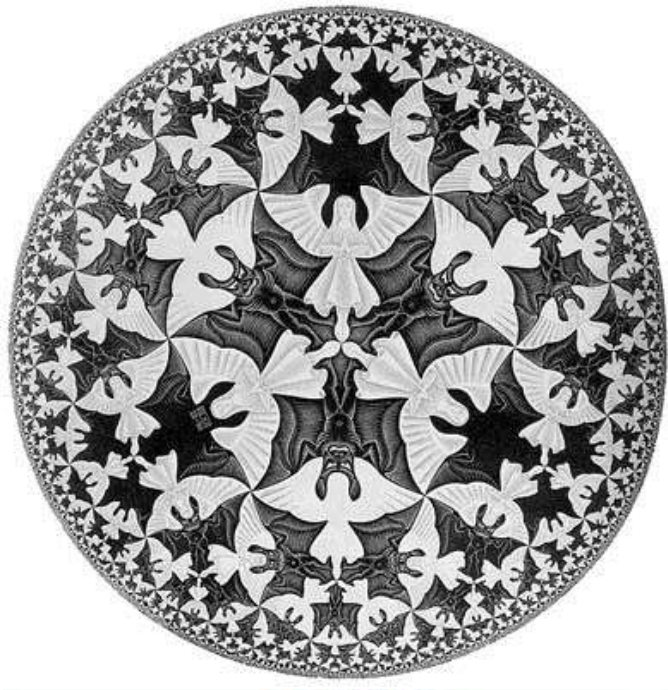
ROCK, CONJUNTO-M Y π

“Por ser un lenguaje, las matemáticas no sólo pueden usarse para informar, sino también para seducir”

Benoit Mandelbrot
(20-XI-1924, 14-X-2010)

Marco Aurelio Alzate Monroy
Universidad Distrital
Francisco José de Caldas
Bogotá

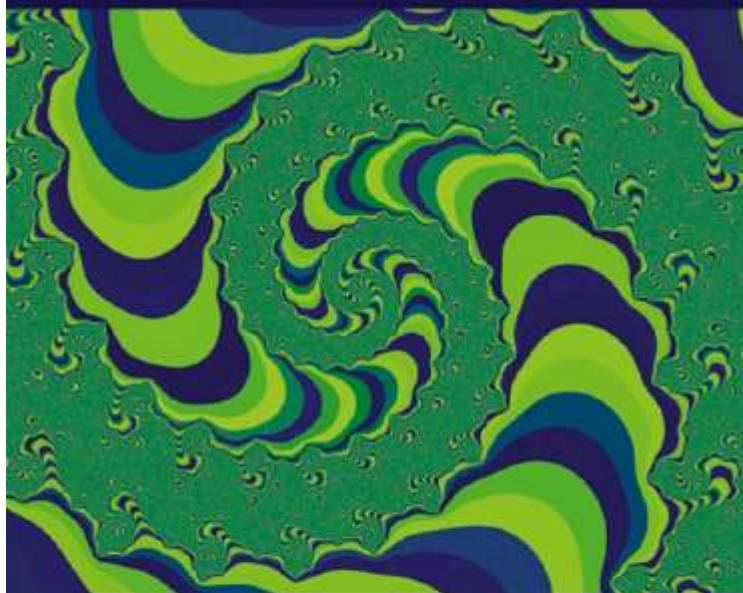




Maurits Cornelis Escher

ELEMENTS OF INFORMATION THEORY

SECOND EDITION



THOMAS M. COVER
JOY A. THOMAS

and

$$\begin{aligned}\frac{\partial^2}{\partial y^2} g_t(y) &= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi t}} \frac{\partial}{\partial y} \left[-\frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}} \right] dx \\ &= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi t}} \left[-\frac{1}{t} e^{-\frac{(y-x)^2}{2t}} + \frac{(y-x)^2}{t^2} e^{-\frac{(y-x)^2}{2t}} \right] dx.\end{aligned}\quad (17.78)$$

$$(17.79)$$

Thus,

$$\frac{\partial}{\partial t} g_t(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} g_t(y). \quad (17.80)$$

We will use this relationship to calculate the derivative of the entropy of Y_t , where the entropy is given by

$$h_e(Y_t) = - \int_{-\infty}^{\infty} g_t(y) \ln g_t(y) dy. \quad (17.81)$$

Differentiating, we obtain

$$\frac{\partial}{\partial t} h_e(Y_t) = - \int_{-\infty}^{\infty} \frac{\partial}{\partial t} g_t(y) dy - \int_{-\infty}^{\infty} \frac{\partial}{\partial t} g_t(y) \ln g_t(y) dy \quad (17.82)$$

$$= - \frac{\partial}{\partial t} \int_{-\infty}^{\infty} g_t(y) dy - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2} g_t(y) \ln g_t(y) dy. \quad (17.83)$$

The first term is zero since $\int g_t(y) dy = 1$. The second term can be integrated by parts to obtain

$$\frac{\partial}{\partial t} h_e(Y_t) = -\frac{1}{2} \left[\frac{\partial g_t(y)}{\partial y} \ln g_t(y) \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial y} g_t(y) \right]^2 \frac{1}{g_t(y)} dy. \quad (17.84)$$

The second term in (17.84) is $\frac{1}{2} J(Y_t)$. So the proof will be complete if we show that the first term in (17.84) is zero. We can rewrite the first term as

$$\frac{\partial g_t(y)}{\partial y} \ln g_t(y) = \left[\frac{\frac{\partial g_t(y)}{\partial y}}{\sqrt{g_t(y)}} \right] \left[2\sqrt{g_t(y)} \ln \sqrt{g_t(y)} \right]. \quad (17.85)$$

Seducción en la cafetería



Fender®

Rock, conjunto de
Mandelbrot y π

Marco Aurelio Alzate Monroy
Universidad Distrital F.J.C.

Starcaster o Stratocaster



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Rock, conjunto de
Mandelbrot y π

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Robert Redford,
Brad Pitt,
Leonardo DiCaprio
y Tom Cruise



Rock, conjunto de
Mandelbrot y π

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Seducción en el bar-rocko

AGENDA



Pathological monsters!
Cried the terrified mathematician
Every one of them is a splinter in my eye
I hate the Peano Space and the Koch Curve
I fear the Cantor Ternary Set
And the Sierpinski Gasket makes me want to cry

And a million miles away a butterfly flapped its wings
On a cold November day
A man named Benoit Mandelbrot was born

His disdain for pure mathematics
And his unique geometrical insights
Left him well equipped to face those demons down
He saw that infinite complexity
Could be described by simple rules
He used his giant brain to turn the game around
And he looked below the storm
And saw a vision in his head, A bulbous pointy form
He picked his pencil up and he wrote his secret down

Just take a point called Z in the complex plane
Let Z_1 be Z squared plus C
And Z_2 is Z_1 squared plus C
And Z_3 is Z_2 squared plus C and so on
If the series of Z 's should always stay close to Z
And never trend away
That point is in the Mandelbrot Set

Mandelbrot Set you're a Rorschach Test on fire
You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire
You're one badass fucking fractal
And you're just in time to save the day

Sweeping all our fears away
You can change the world in a tiny way

Mandelbrot's in heaven, at least he will be when he's dead
Right now he's still alive and teaching math at Yale
He gave us order out of chaos
He gave us hope where there was none
And his geometry succeeds where others fail
If you ever lose your way, a butterfly will flap its wings
From a million miles away, a little miracle will come
To take you home

Just take a point called Z in the complex plane
Let Z_1 be Z squared plus C
And Z_2 is Z_1 squared plus C
And Z_3 is Z_2 squared plus C and so on
If the series of Z 's should always stay close to Z
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You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire
You're one badass fucking fractal
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
Go on change the world in a tiny way
Come on change the world in a tiny way

Mandelbrot Set
by Jonathan Coulton

PRIMERA PARTE

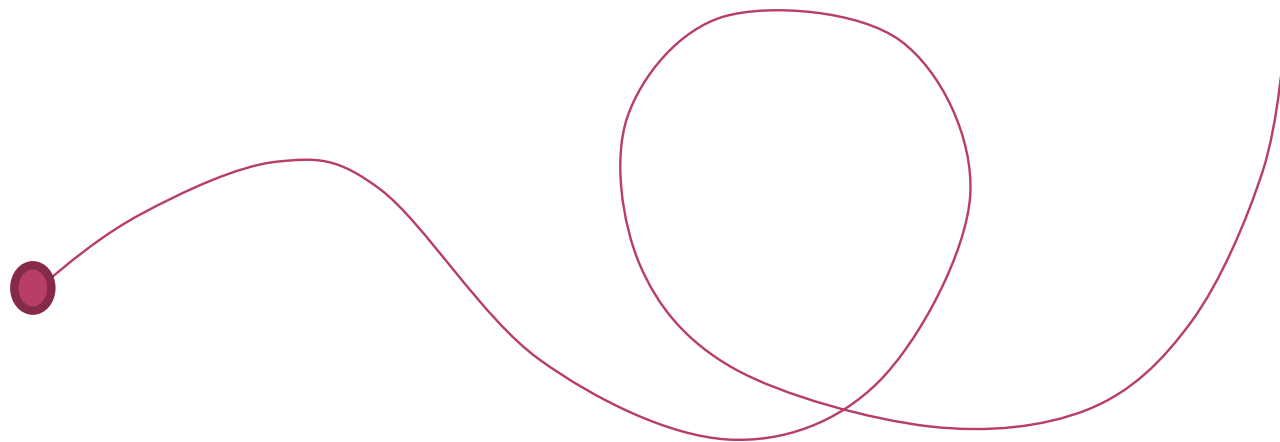
⦿ Pathological monsters!

**Cried the terrified mathematician
Every one of them is a splinter in my eye
I hate the Peano Space and the Koch Curve
I fear the Cantor Ternary Set
And the Sierpinski Gasket makes me
want to cry**

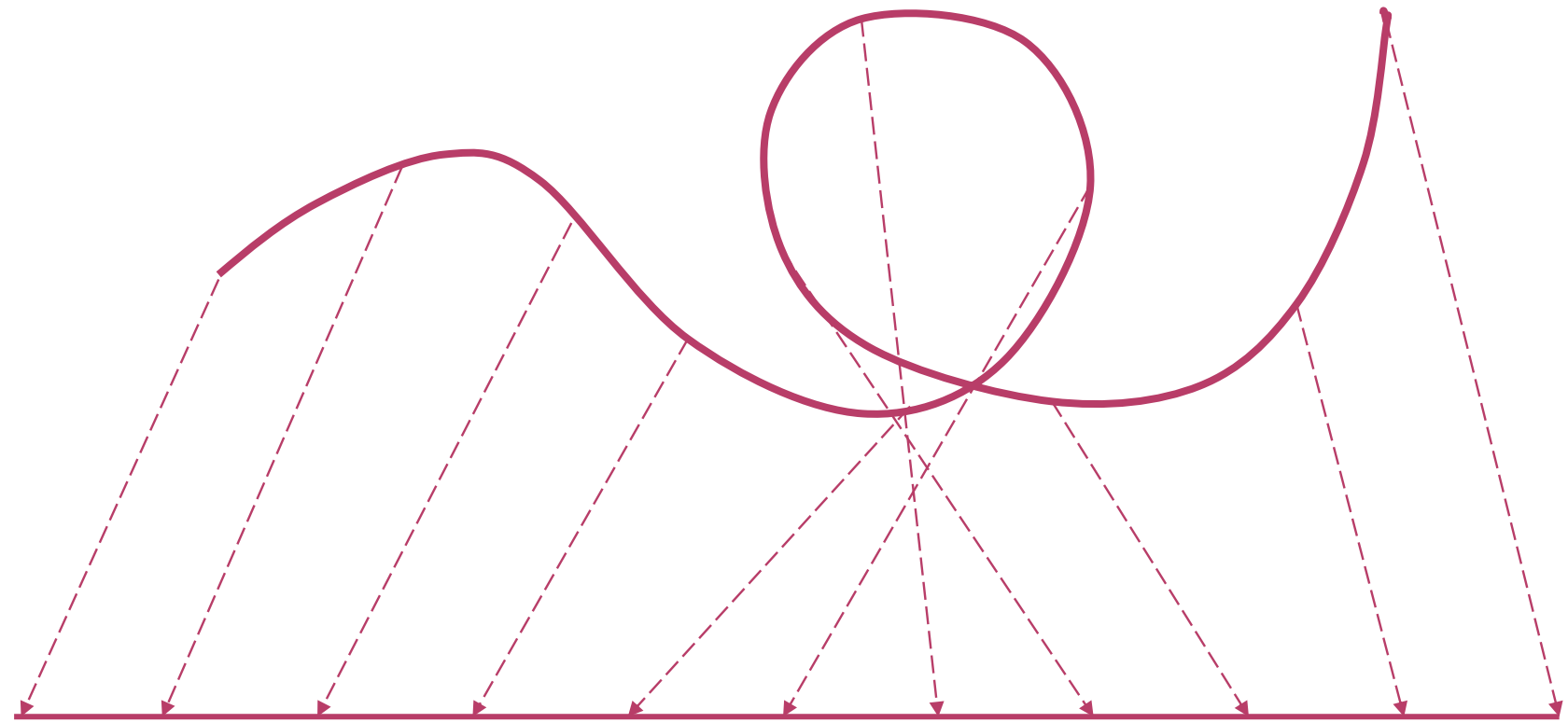
“MONSTRUOS PATOLOGICOS!” GRITÓ EL MATEMÁTICO ATERRORIZADO. “CADA UNO DE ELLOS ES UNA ASTILLA EN MI OJO”...

- ◉ Weierstrass, Cantor, Poincare, Peano, Hilbert, Von Koch, Sierpinski, Hausdorff, Julia, Fatou, ... Matemáticos que supieron sobre fractales antes de Mandelbrot, pero nunca los pudieron ver.
- ◉ En varios de ellos, el trabajo que adelantaban produjo serias consecuencias en su salud mental
- ◉ “Monstruos matemáticos”
“Ecuaciones patológicas”

EL ESPACIO DE PEANO

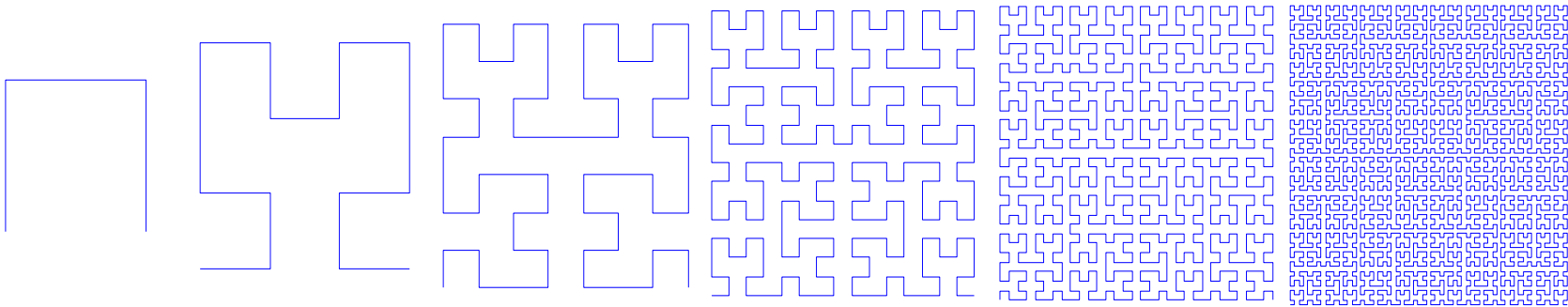


EL ESPACIO DE PEANO



La trayectoria de una partícula en un espacio n -dimensional siempre podrá trasladarse a una sola dimensión

¿QUÉ TAL ESTA SECUENCIA DE TRAYECTORIAS?



```
function Espacio_de_Peano
    for i=1:6
        subplot(1,6,i)
        [x,y] = iteracion_hilbert(i);
        plot(x,y)
    end
```

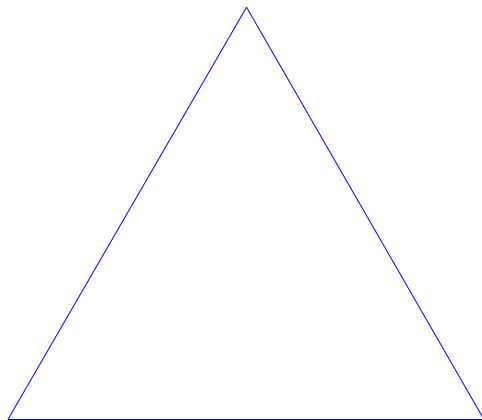
```
function [x,y] = iteracion_hilbert(n)
    if n>0
        [a,b]=iteracion_hilbert(n-1);
        x=[-.5+b -.5+a .5+a .5-b]/2;
        y=[-.5+a .5+b .5+b -.5-a]/2;
    else
        x=0; y=0;
    end
```

Aunque es una trayectoria de una partícula, en un número infinito contable de iteraciones logra recorrer todos los puntos (infinitos no contables) de un plano... **Bi-dimensional!**

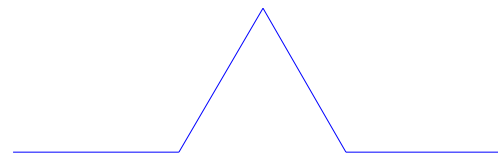
Curva de Hilbert

SISTEMAS-L: RE-ESCRITURA

- Re-escritura: define objetos complejos remplazando sucesivamente partes de un objeto inicial simple mediante un conjunto de reglas de re-escritura (Gramática formal de Chomski, 1950, equivalente a la re-escritura algol de Backus-Naur, 1960).
- El ejemplo clásico es el copo de nieve de Koch, propuesto en 1905 por Helge Von Koch, al que Mandelbrot se refiere así:
 - Se empieza con dos formas, una iniciadora y una generadora. La segunda es una línea dirigida, hecha por N lados iguales de longitud r . Cada etapa de la construcción empieza con una línea quebrada y reemplaza cada intervalo recto con una copia de la forma generadora, reducida y desplazada de manera que tenga los mismos puntos extremos de la línea original.

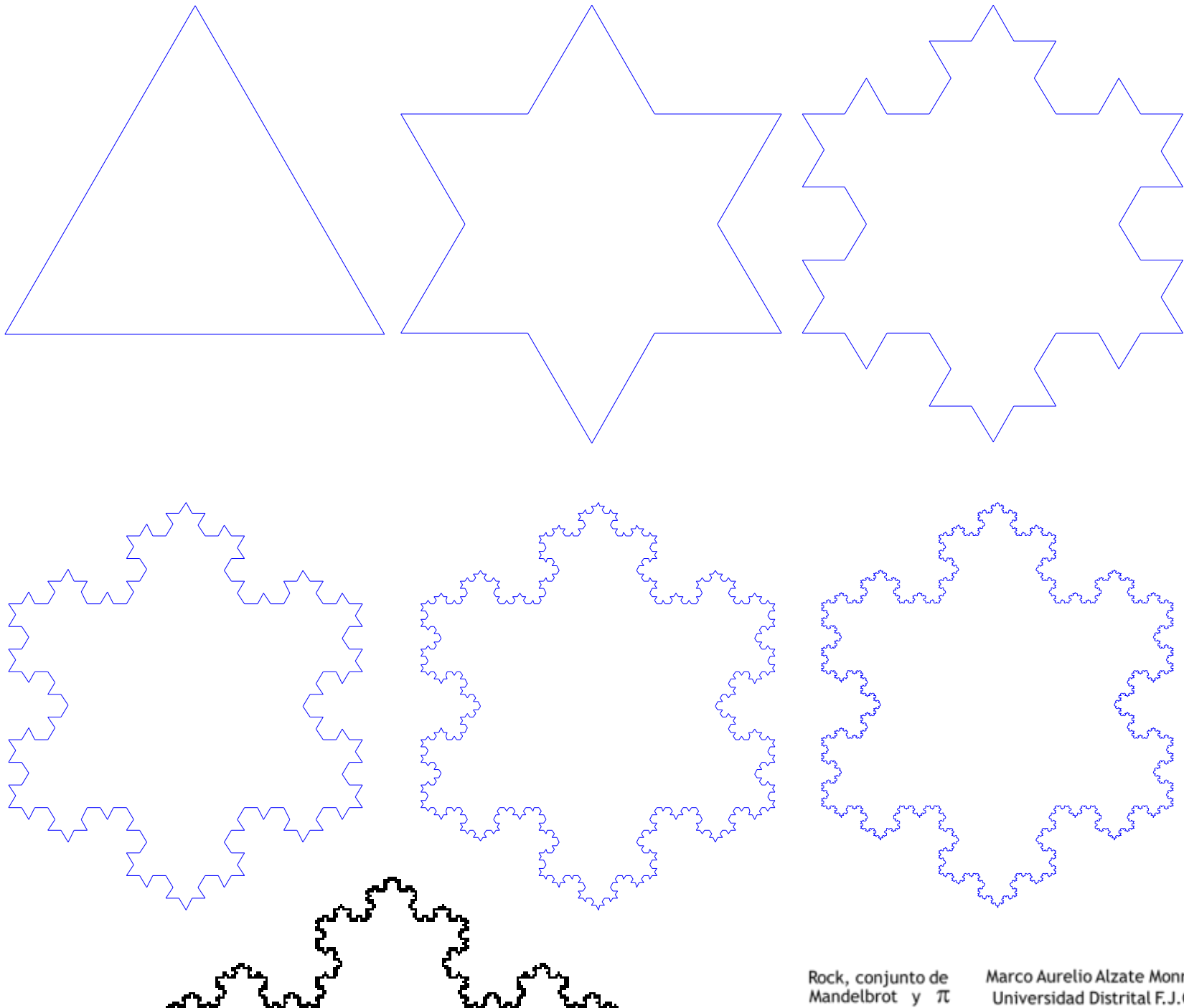


Iniciador



Generador

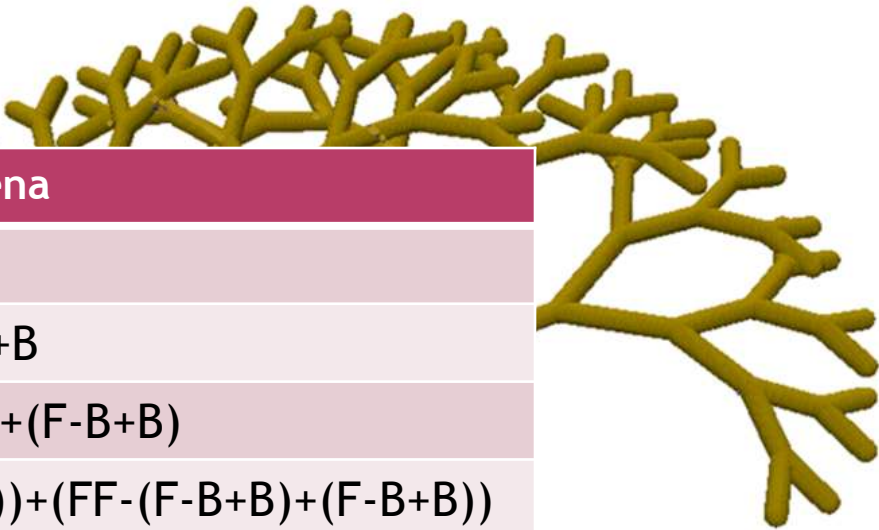
And the Koch curve



SISTEMA-L

Axiom : B

Rules : $B \rightarrow F-B+B$, $F \rightarrow FF$



Koch curve

Paso	Cadena
0	B
1	F-B+B
2	FF-(F-B+B)+(F-B+B)
3	FFFF-(FF-(F-B+B)+(F-B+B))+(FF-(F-B+B)+(F-B+B))

“La idea de la forma también contiene, implícitamente, la historia de dicha forma”.

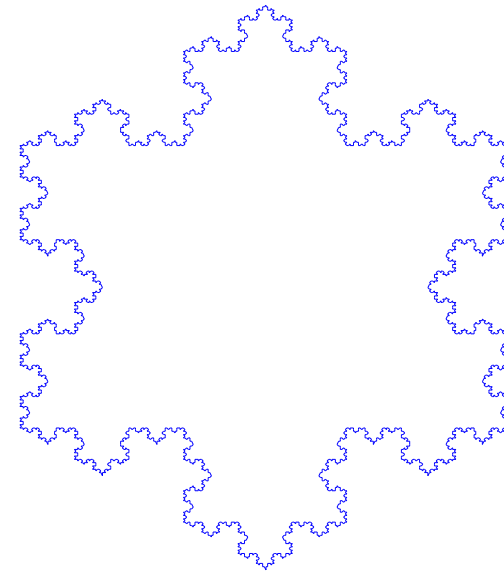
A. Lindenmayer



UN SISTEMA “CASI-L”

```
function Tortuga(delta,Axioma)
    x0 = 0; y0 = 0; distancia = 10; angulo = 0;
    i = 1;
    clf
    while Axioma(i)~='.'
        switch Axioma(i)
            case 'f'
                x1 = x0 + distancia*cos(angulo);
                y1 = y0 + distancia*sin(angulo);
                plot([x0 x1],[y0 y1])
                hold on
                x0 = x1; y0 = y1;
            case 'l'
                angulo = angulo + delta;
            case 'r'
                angulo = angulo - delta;
        end
        i = i+1;
    end
```

```
function secuencia(delta,Generador,Axioma)
    for n=1:5
        SistemaCasiL(delta,Generador,Axioma,n);
        drawnow
        pause(1)
    end
```

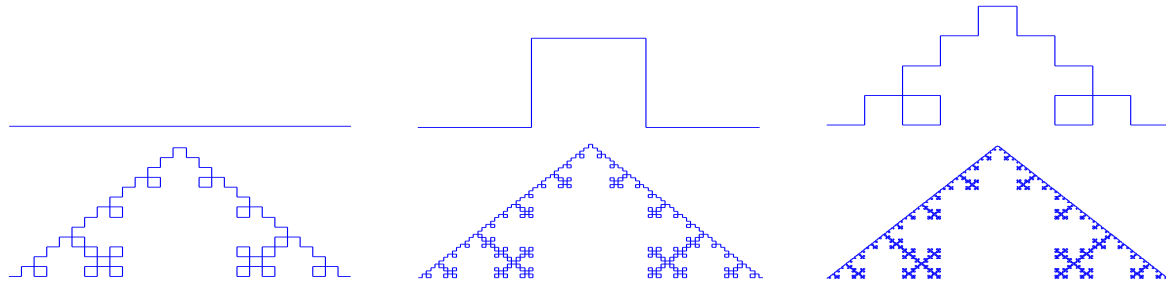


```
function SistemaCasiL(delta,Generador,Axioma,n)
    while n>1
        buffer = '';
        for i=1:length(Axioma)
            if Axioma(i) == 'f'
                buffer = [buffer Generador];
            else
                buffer = [buffer Axioma(i)];
            end
        end
        Axioma = buffer;
        n = n-1;
    end
    Axioma = [Axioma '.'];
    Tortuga(delta,Axioma)
```

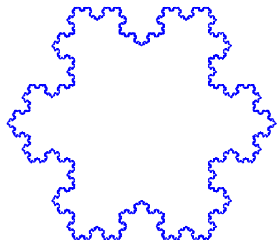
And the Koch curve

ALGUNOS FRACTALES CONSEGUIDOS CON EL ALGORITMO DE UNA SOLA TORTUGA

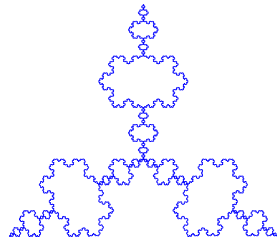
$\text{delta} = \pi/2;$
Generador = 'flfrfrflf';
Axioma = 'f';



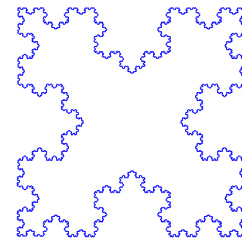
$\text{delta} = \pi/3;$
Generador = 'flfrrflf';
Axioma = 'flflflflflfl';



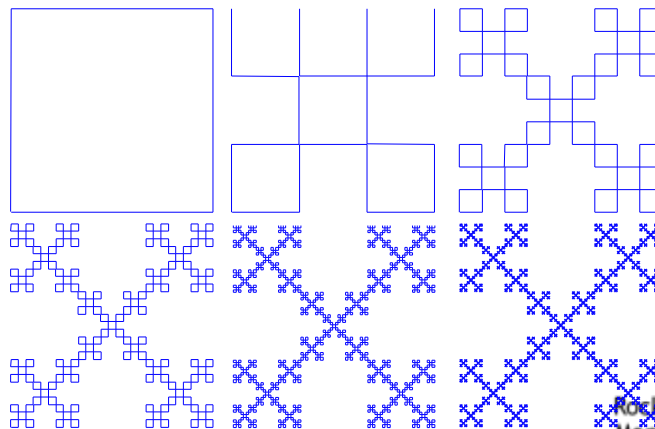
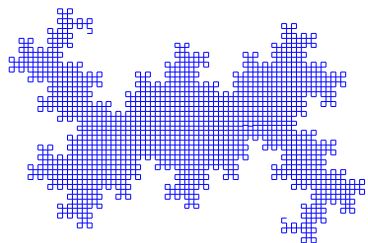
$\text{delta} = \pi/3;$
Generador = 'flfrrflf';
Axioma = 'fllflflfll';



$\text{delta} = \pi/6;$
Generador = 'fllfrrrrfllf';
Axioma = 'flllflllflllflll';



$\text{delta} = \pi/2;$
Generador = 'flflfrfrf';
Axioma = 'f';



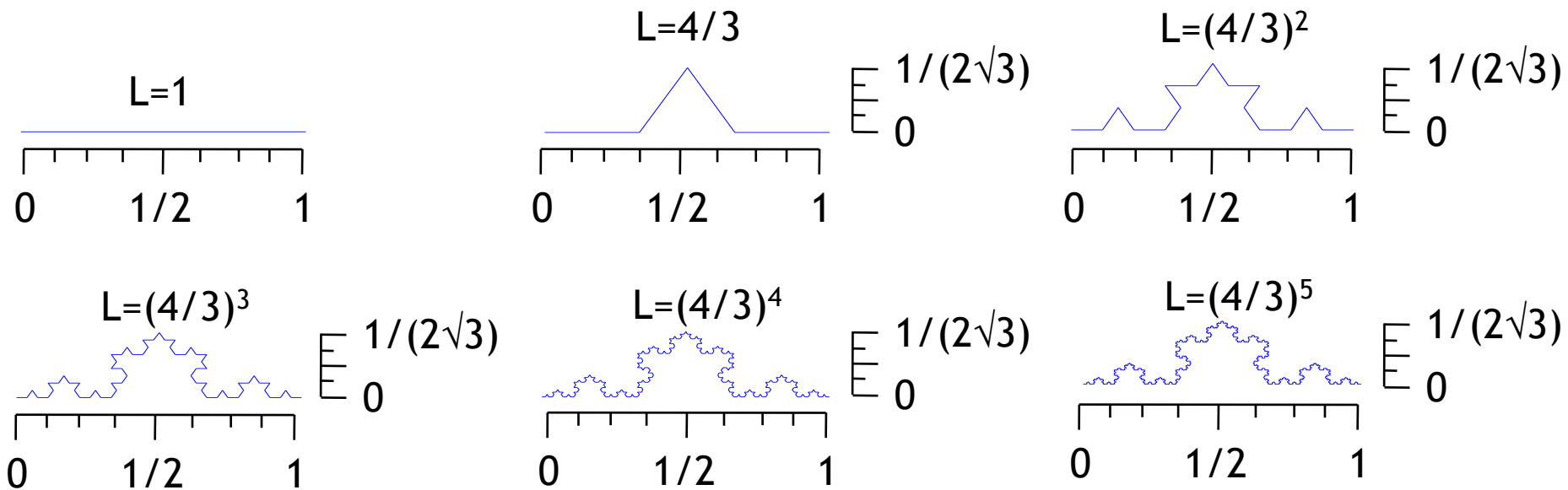
$\text{delta} = \pi/2;$
Generador = 'flfrfrflf';
Axioma = 'flflflfl';

Rock, conjunto de
Mandelbrot y π

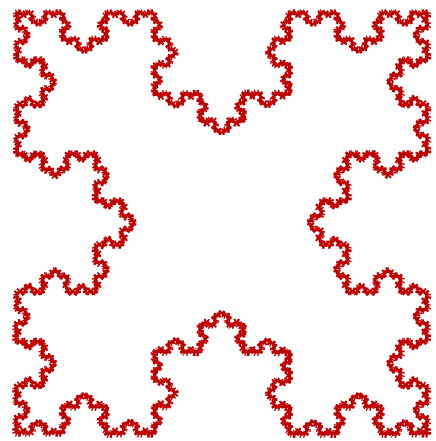
Marco Aurelio Alzate Monroy
 Universidad Distrital F.J.C.

And the Koch curve

¿QUÉ LONGITUD TIENE LA CURVA DE KOCH?



La longitud es $\lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n = \infty$ Pero el área bajo ella es $\sum_{n=1}^{\infty} \left(4^{n-1}\right) \left(\frac{\sqrt{3}}{4} 9^{-n}\right) = \frac{\sqrt{3}}{20}$

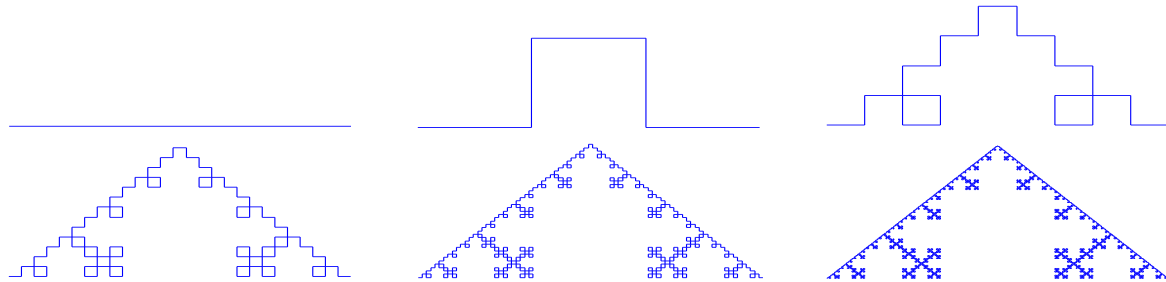


**Esta figura tiene perímetro infinito,
pero un área de sólo $\frac{5 - \sqrt{3}}{5} \text{ m}^2$!**

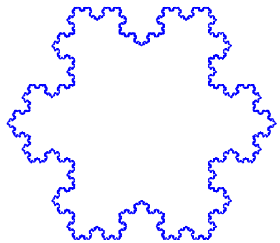
Además, es continua en todo punto pero no derivable en ninguno!

ALGUNOS FRACTALES CONSEGUIDOS CON EL ALGORITMO DE UNA SOLA TORTUGA

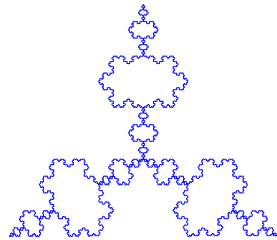
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Generador = 'flfrfrflf';
Axioma = 'f';



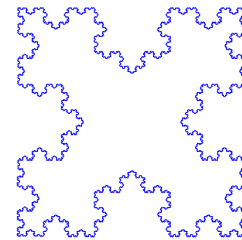
delta = pi/3;
Generador = 'flfrrflf';
Axioma = 'flflflflflfl';



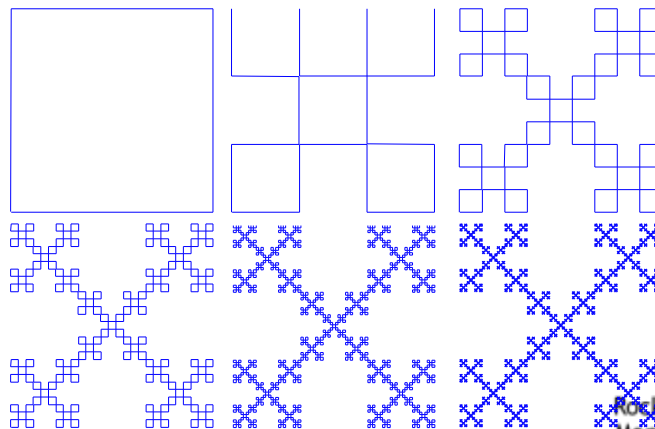
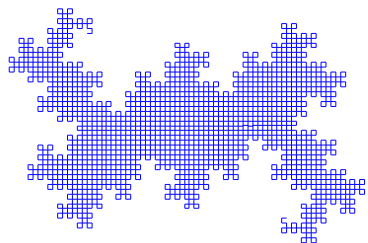
delta = pi/3;
Generador = 'flfrrflf';
Axioma = 'fllflflfll';



delta = pi/6;
Generador = 'fllfrrrrfllf';
Axioma = 'flllflllflllflll';



delta = pi/2;
Generador = 'flflfrfrf';
Axioma = 'f';



delta = pi/2;
Generador = 'flfrfrflf';
Axioma = 'flflflfl';

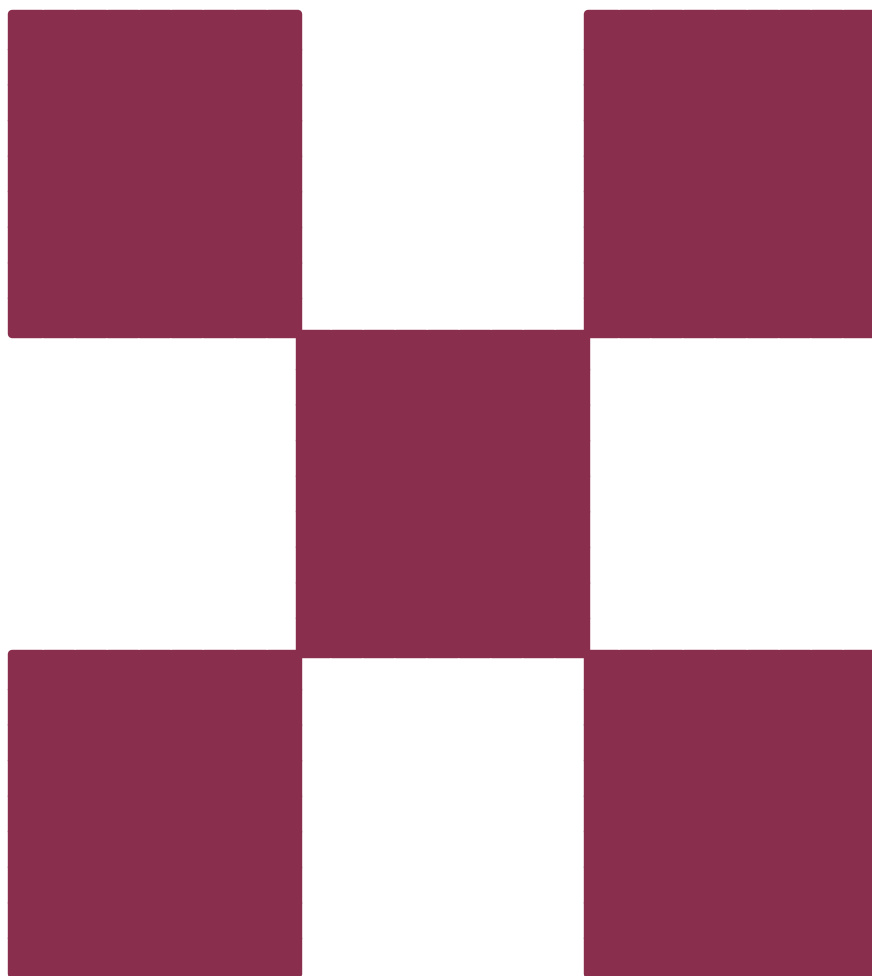
Rock, conjunto de
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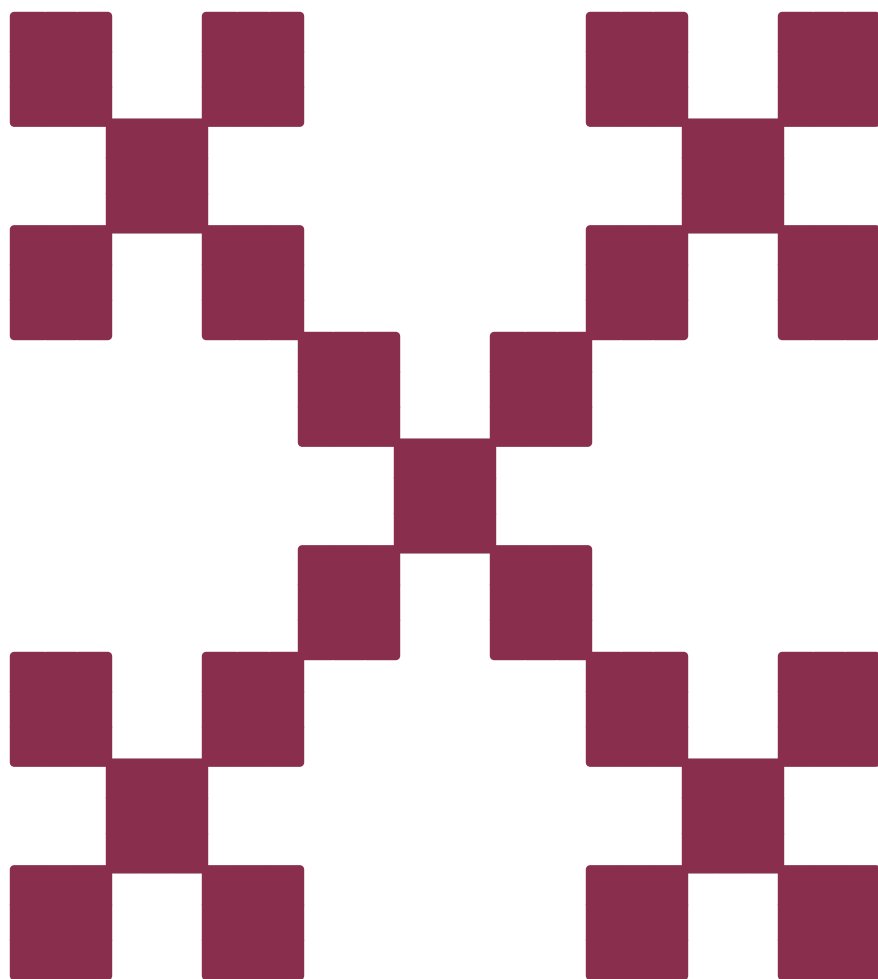
And the Koch curve



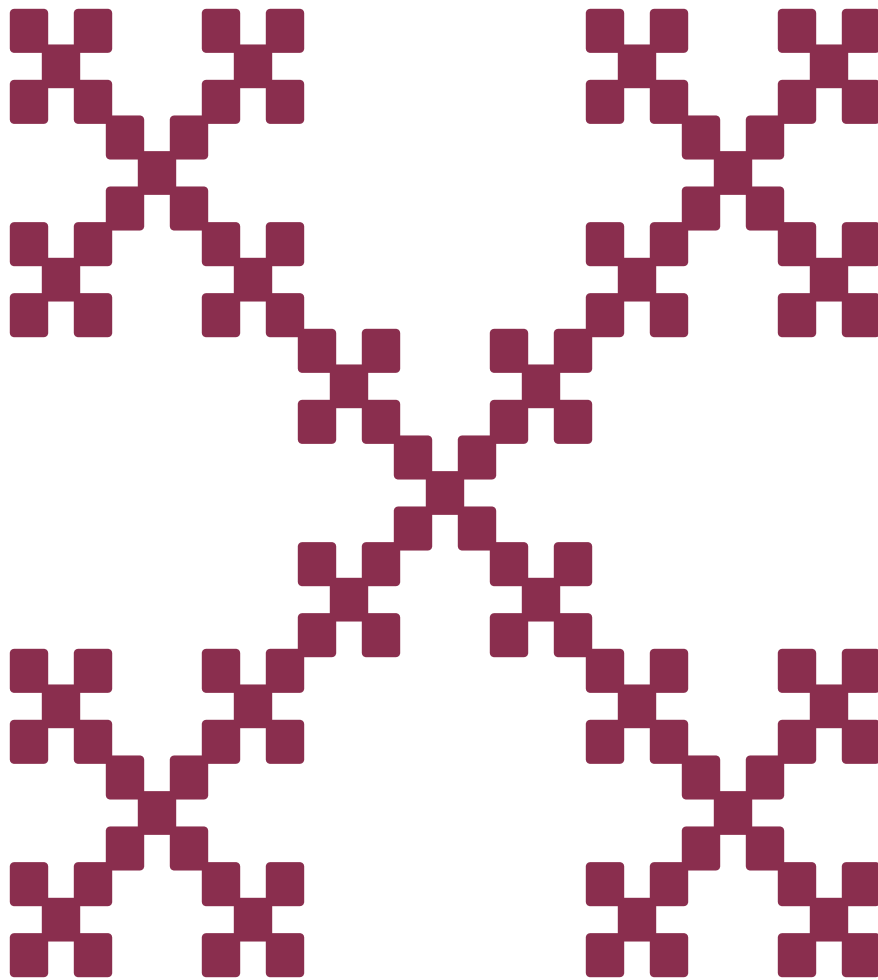
OTRO ALGORITMO PARA
CONSTRUIR EL MISMO FRACTAL



OTRO ALGORITMO PARA
CONSTRUIR EL MISMO FRACTAL

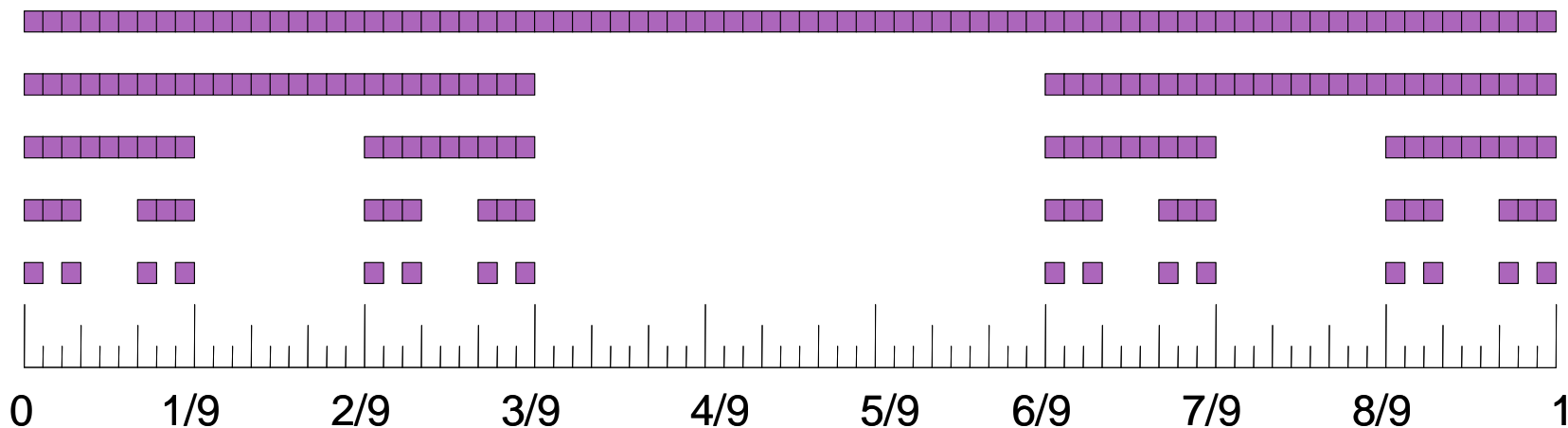


OTRO ALGORITMO PARA
CONSTRUIR EL MISMO FRACTAL



OTRO ALGORITMO PARA
CONSTRUIR EL MISMO FRACTAL

EL CONJUNTO DE CANTOR



¿Cuántos intervalos hay en el conjunto de Cantor?

En el n -ésimo paso retiramos 2^n intervalos, cada uno de longitud $3^{-(n+1)}$

Longitud total de los intervalos retirados:

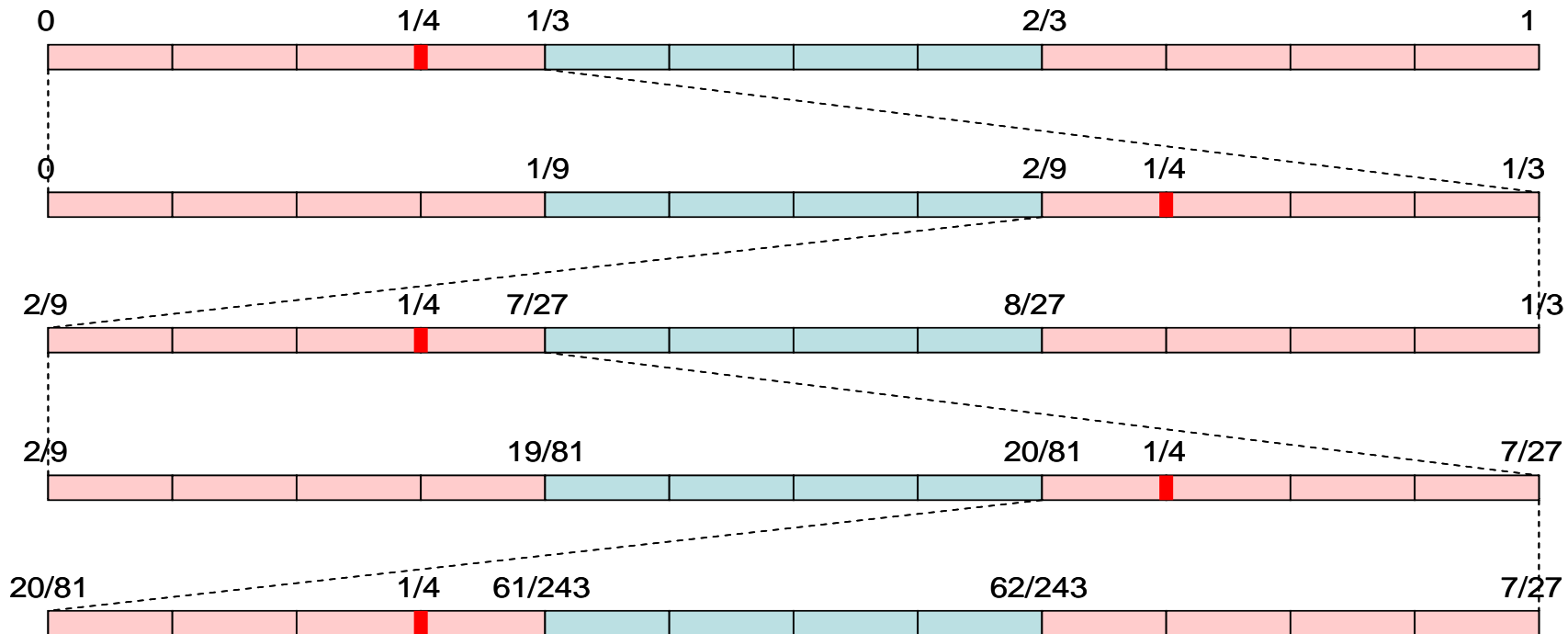
$$\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n = \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}} \right) = 1$$

Si el conjunto de Cantor es un conjunto de puntos aislados, ¿Podemos contarlos?

I fear the Cantor ternary set



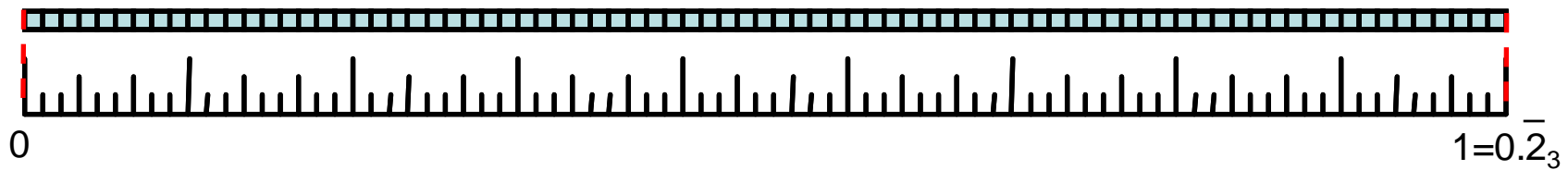
No es tan simple como contar los extremos de los intervalos que van quedando en cada paso, porque hay muchos puntos intermedios que jamás se retiran



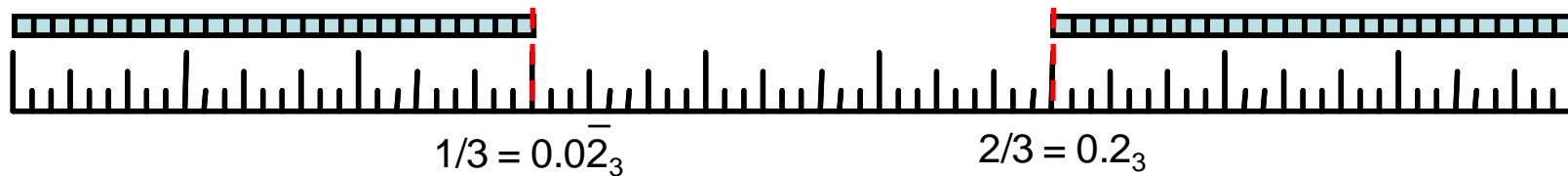
El punto $\frac{1}{4}$, por ejemplo, no es el extremo de ningún intervalo pero jamás se retira del conjunto de Cantor!



Más fácil si usamos numeración en base 3:



$$0.\overline{2}_3 = \sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{2}{3} \left(\frac{1}{1 - \frac{1}{3}}\right) = 1$$



¿QUÉ PUNTOS PERTENECEN AL CONJUNTO DE CANTOR?



I fear the Cantor ternary set

$0.0XX_3$

$0.1XX_3$

$0.2XX_3$



$0.00X_3$ $0.01X_3$ $0.02X_3$



$0.20X_3$ $0.21X_3$ $0.22X_3$



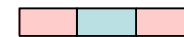
$0.000X_3$
 $0.001X_3$
 $0.002X_3$



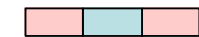
$0.020X_3$
 $0.021X_3$
 $0.022X_3$



$0.200X_3$
 $0.201X_3$
 $0.202X_3$



$0.220X_3$
 $0.221X_3$
 $0.222X_3$



El conjunto de Cantor consiste en aquellos puntos $c \in [0, 1]$ que no tienen 1's en su representación base-3

No sólo los puntos extremos de los intervalos ternarios:

$$0.\overline{02}_3 = 2 \sum_{n=1}^{\infty} \frac{1}{3^{2n}} = \frac{2}{9} \sum_{n=0}^{\infty} \left(\frac{1}{9} \right)^n = \frac{2}{9} \left(\frac{1}{1 - \frac{1}{9}} \right) = \frac{1}{4}$$

¿CUÁNTOS PUNTOS HAY EN EL CONJUNTO DE CANTOR?



- ◉ El conjunto de Cantor consiste en aquellos puntos $c \in [0, 1]$ que no tienen 1's en su representación base-3, de manera que sólo tienen 0's and 2's.
 - ◉ Cambiando cada 2 en la expansión base-3 de los elementos del conjunto de Cantor por un 1, obtenemos todos los números en el intervalo $[0, 1]$, expresados en binario.
 - ◉ Esta es, exactamente, una representación bi-unívoca entre el conjunto de Cantor y el intervalo de la recta real $[0, 1]$!
 - ◉ Cantor encontró un conjunto de PUNTOS AISLADOS en el intervalo $[0, 1]$ que tiene tantos elementos como el intervalo CONTINUO completo!
-
- ... Obviamente, Cantor se volvió loco! Era 1900, Mandelbrot no había llegado, y los fractales eran “mathematical monsters”

I fear the Cantor ternary set

TRIÁNGULO DE SIERPINSKI



Suponiendo un área inicial de 1, en el paso n retiramos 3^{n-1} triángulos de área 4^{-n} , de manera que, en total, retiramos

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^n} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{4} \left(\frac{1}{1-\frac{3}{4}}\right) = 1$$

Suponiendo un perímetro inicial de 1, en el paso n aumentamos 3^{n-1} triángulos con perímetro 2^{-n} , de manera que, en total, el perímetro es

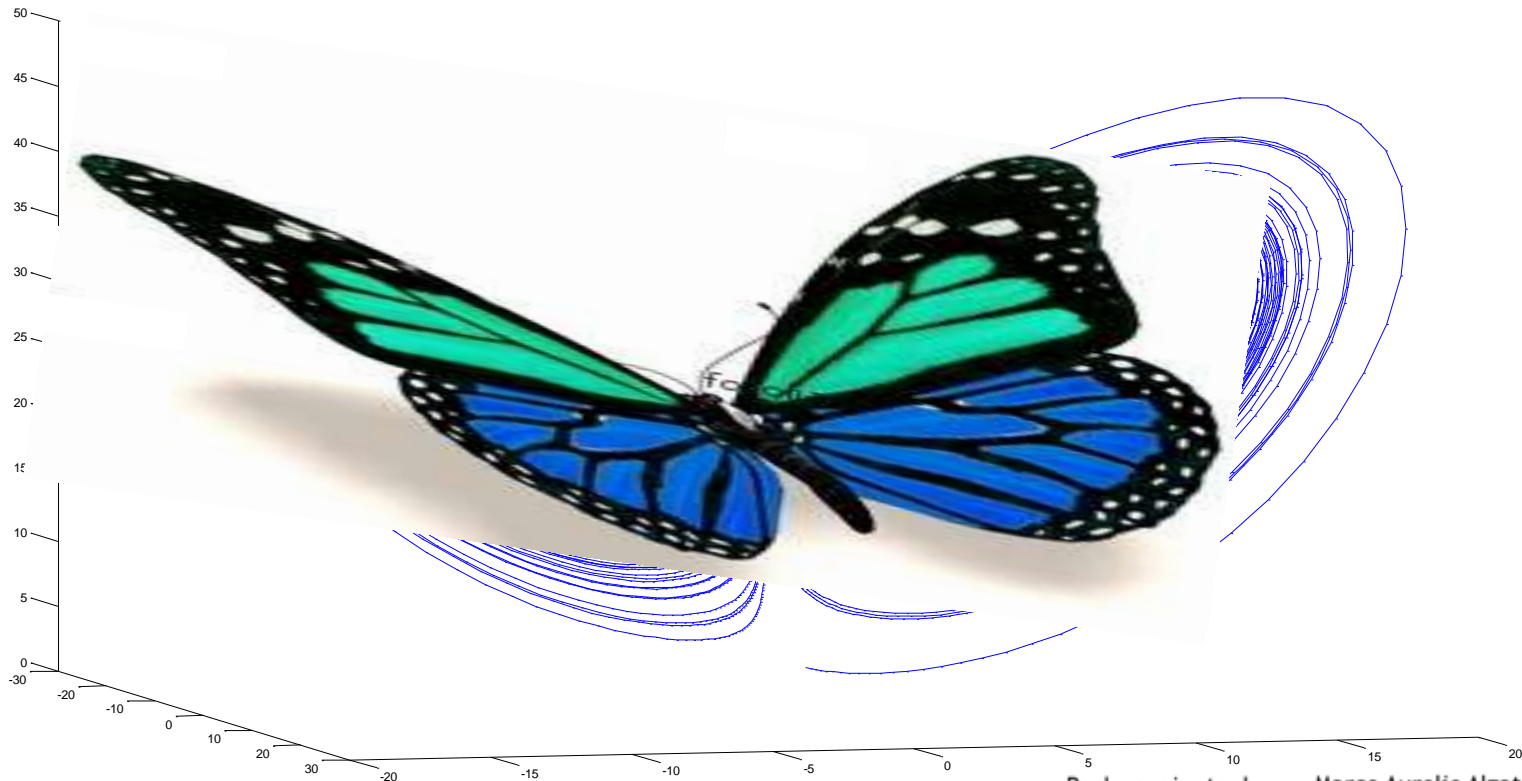
$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = \infty$$

And the Sierpinski Gasket
makes me want to cry

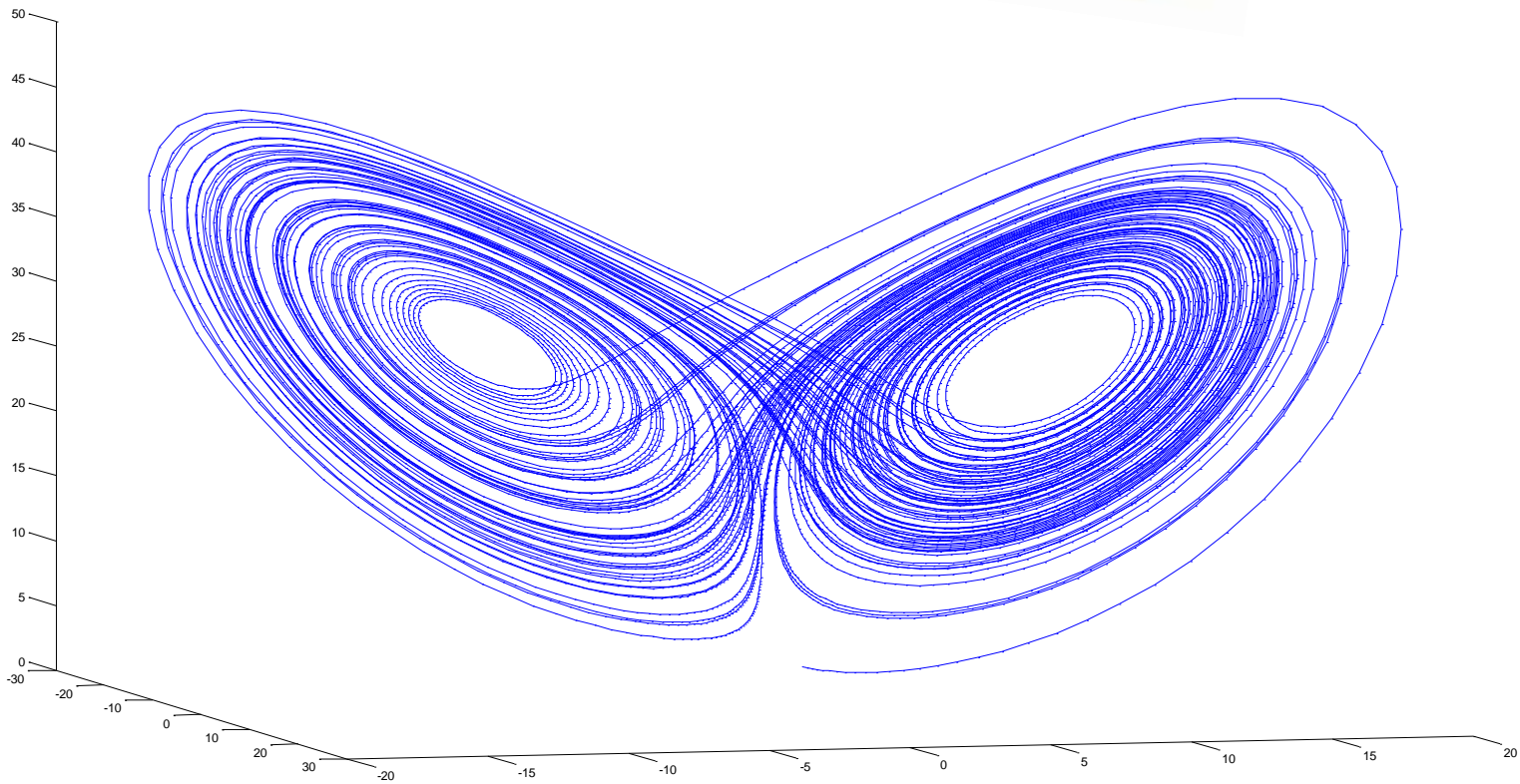
El triángulo de Sierpinski es una línea de longitud infinita que encierra un área cero

SEGUNDA PARTE

- And a million miles away a butterfly flapped its wings

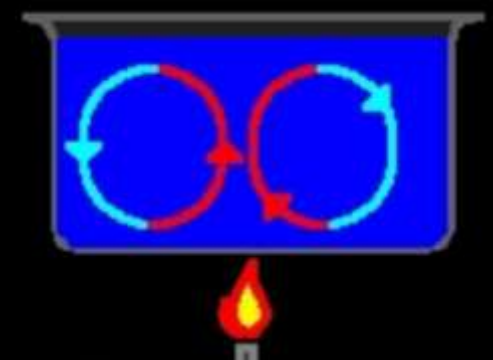


A butterfly flapped its wings



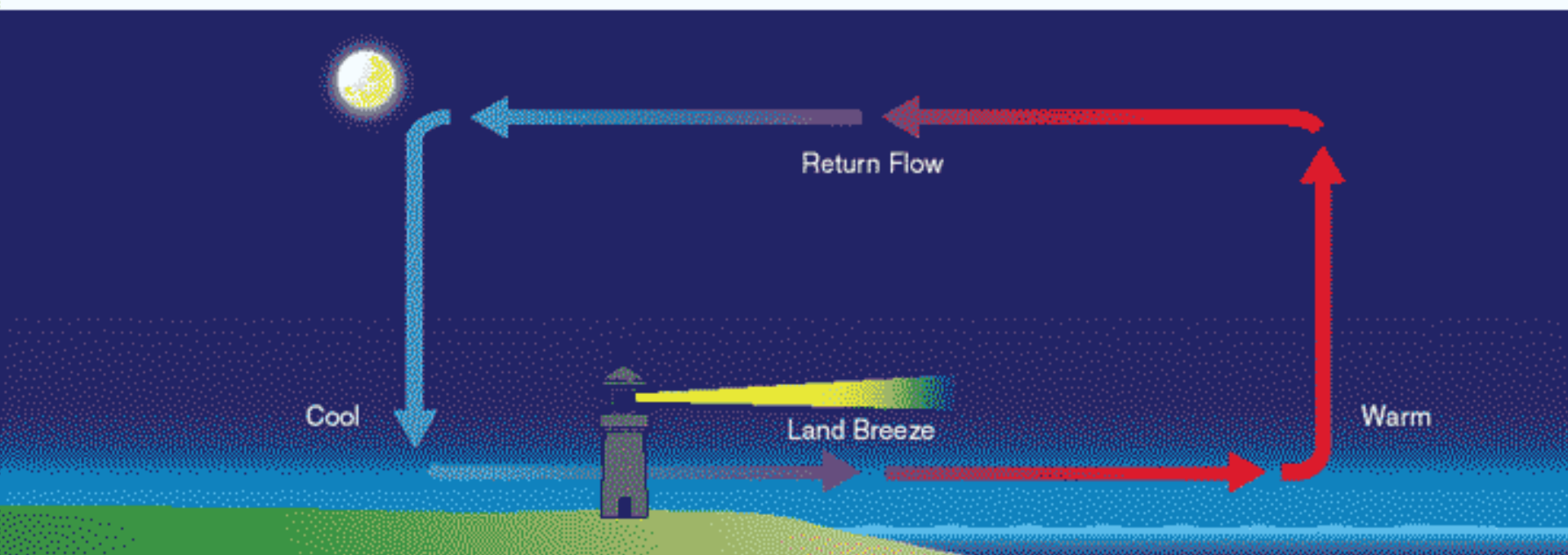
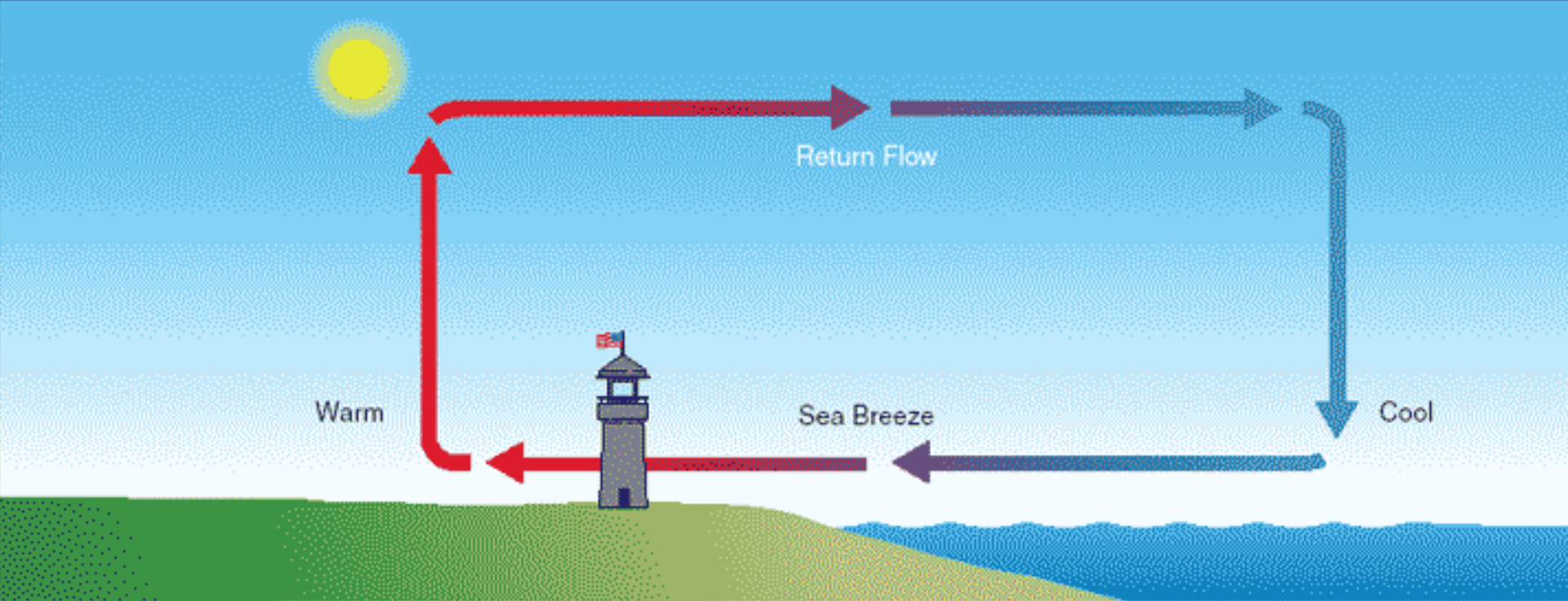
Rock, conjunto de
Mandelbrot y π

Marco Aurelio Alzate Monroy
Universidad Distrital F.J.C.



**Corrientes
de
Convección**

Las corrientes de convección en el manto de la tierra producen el desplazamiento tectónico





Deterministic Nonperiodic Flow¹

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

Fundación de la teoría del caos

Como no se le ocurría un título para una conferencia que iba a dictar en 1972 en el 139 congreso de la Asociación Americana para el Avance de la Ciencia, Philip Merilees, el organizador, lo presentó así: “*Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?*”.

El efecto mariposa
Fundación de la
Fascinación del
Público con la
teoría del caos

ECUACIONES DE LORENZ

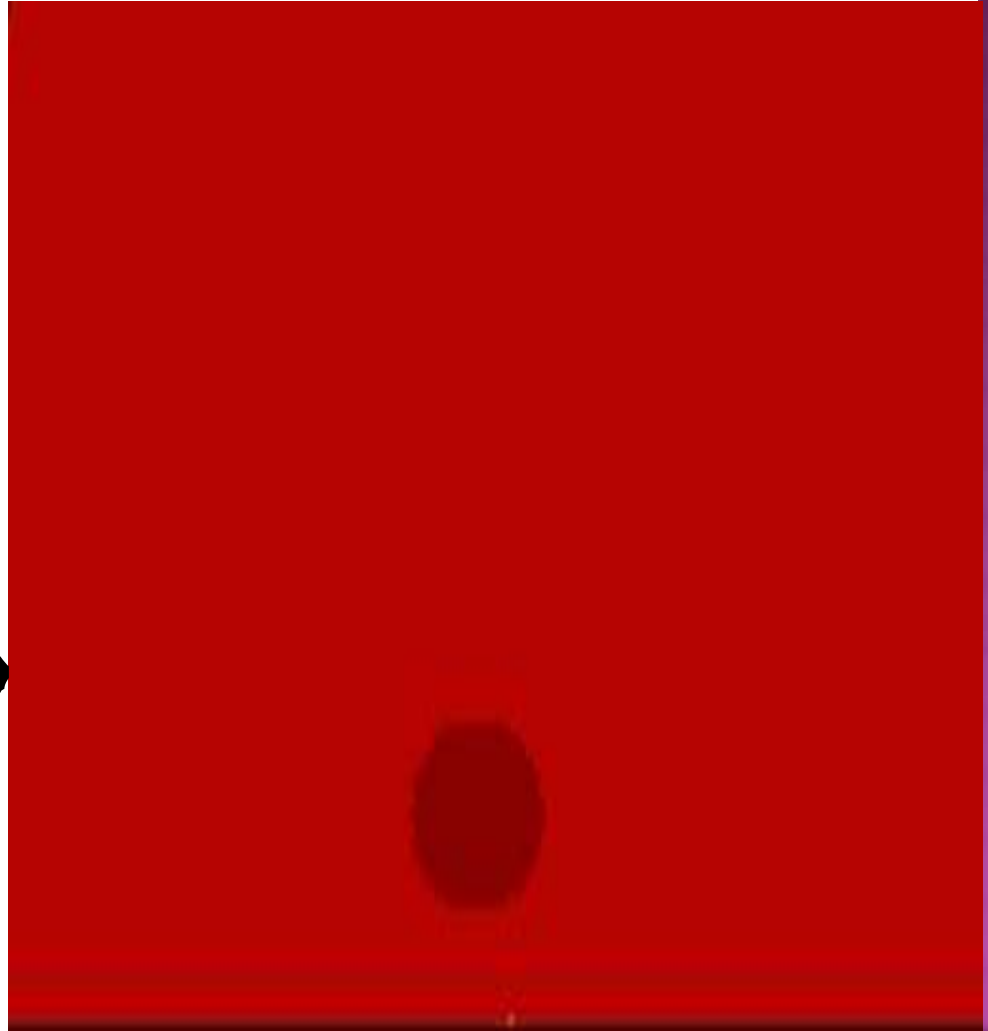
$$\frac{dx}{dt} = ay - ax$$

$$\frac{dy}{dt} = bx - y - zx$$

$$\frac{dz}{dt} = xy - cz$$

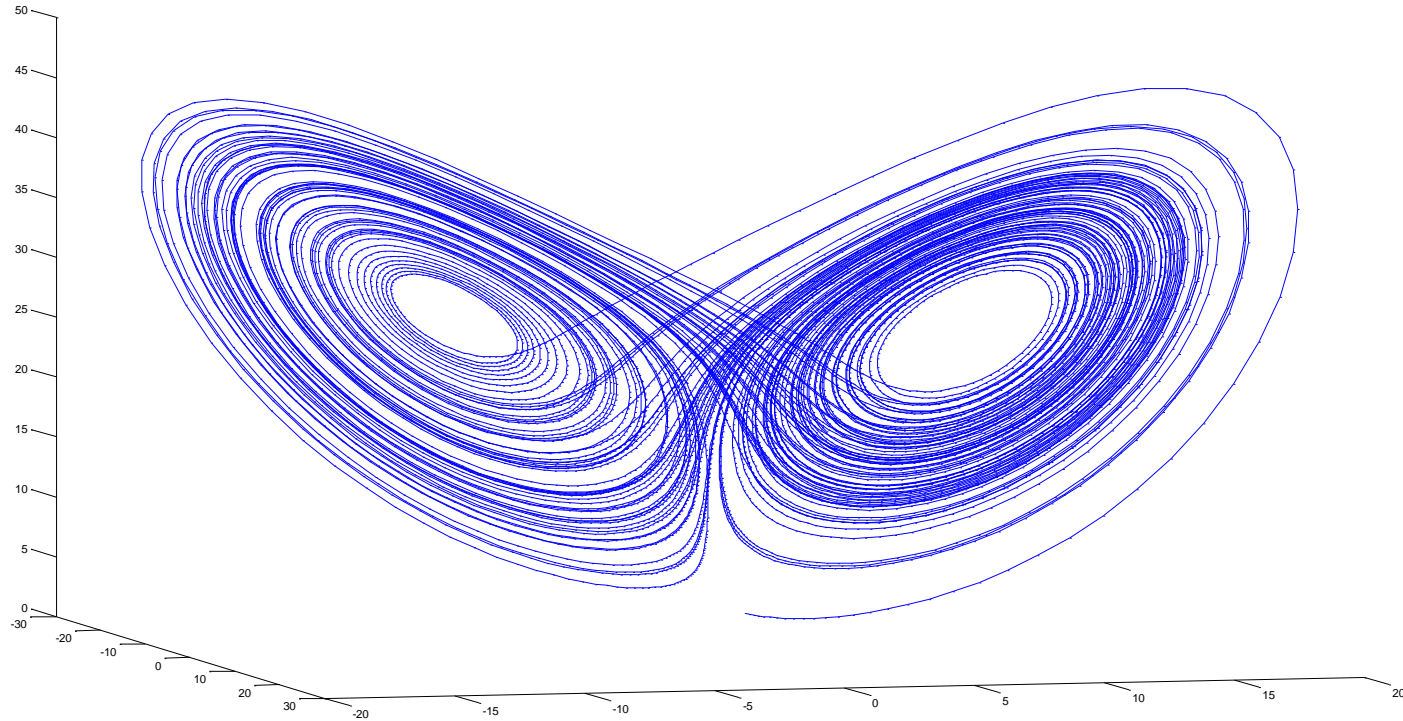
i t : tiempo
 $x(t)$: Posición
 $y(t)$: temperatura
 $z(t)$: “distorsión de T”
 a, b, c : Parámetros

?



A butterfly flapped its wings

ATRACTOR DE LORENZ



```
function x = AtractorDeLorenz
    x(:,1) = [1 2 3]';
    for j=2:10000
        k1 = 0.01*ecuacion(x(:,j-1));
        k2 = 0.01*ecuacion(x(:,j-1) + k1/2);
        k3 = 0.01*ecuacion(x(:,j-1) + k2/2);
        k4 = 0.01*ecuacion(x(:,j-1) + k3);
        x(:,j) = x(:,j-1) + (k1 + 2*k2 + 2*k3 + k4)/6;
    end
    x = x';
    plot3(x(:,1),x(:,2),x(:,3));
```

```
function x = ecuacion(x)
    x = [10*(x(2)-x(1)); 28*x(1)-x(2)-x(1)*x(3); x(1)*x(2)-(8/3)*x(3)];
```

TERCERA PARTE

- On a cold November day
A man named Benoit
Mandelbrot was born



Rock, conjunto de
Mandelbrot y π

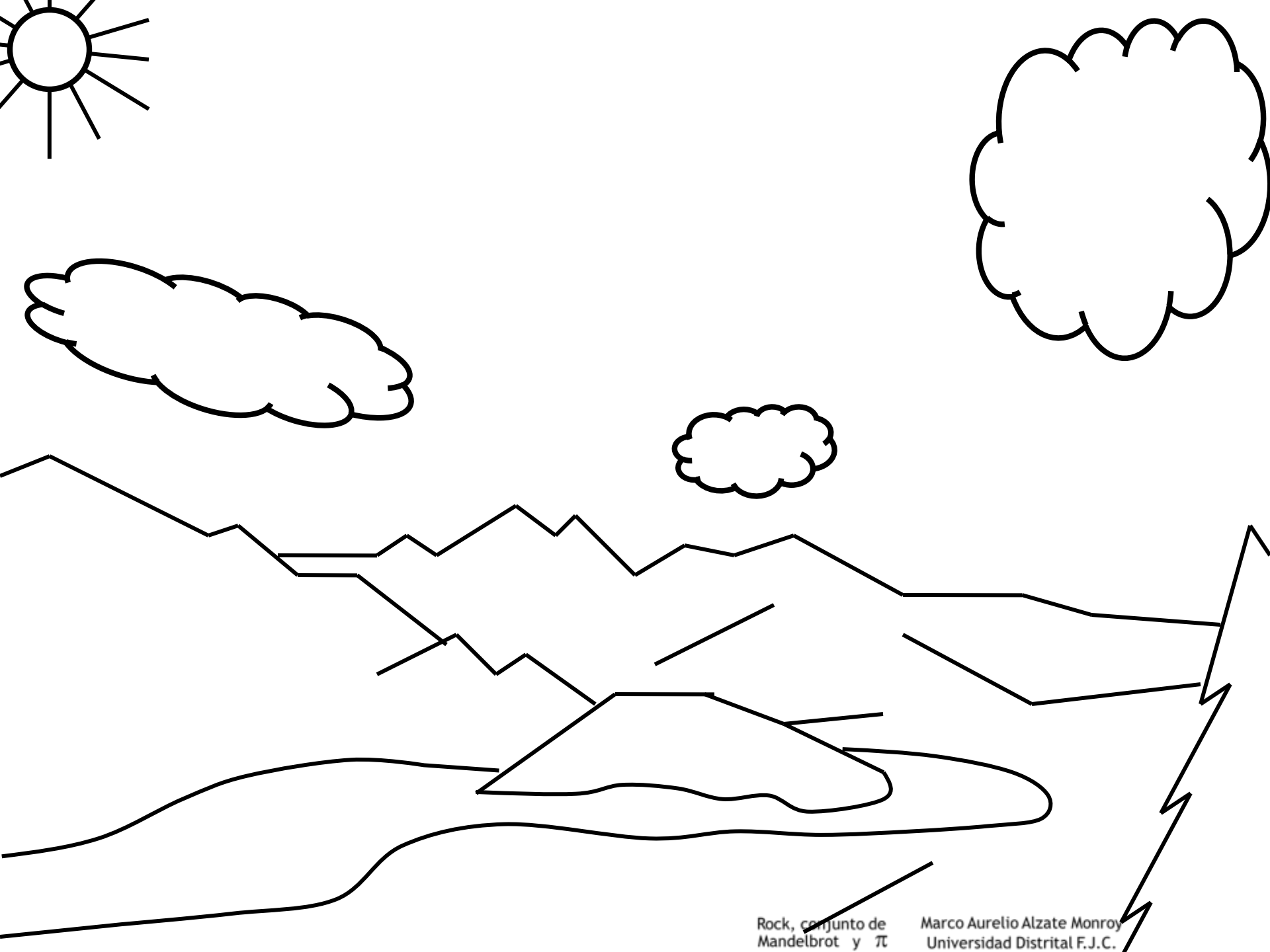
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Universidad Distrital F.J.C.

CUARTA PARTE

- ◉ His disdain for pure mathematics
- ◉ And his unique geometrical insights
- ◉ Left him well equipped to face those demons down



Face those demons down



Rock, conjunto de
Mandelbrot y π

Marco Aurelio Alzate Monroy
Universidad Distrital F.J.C.

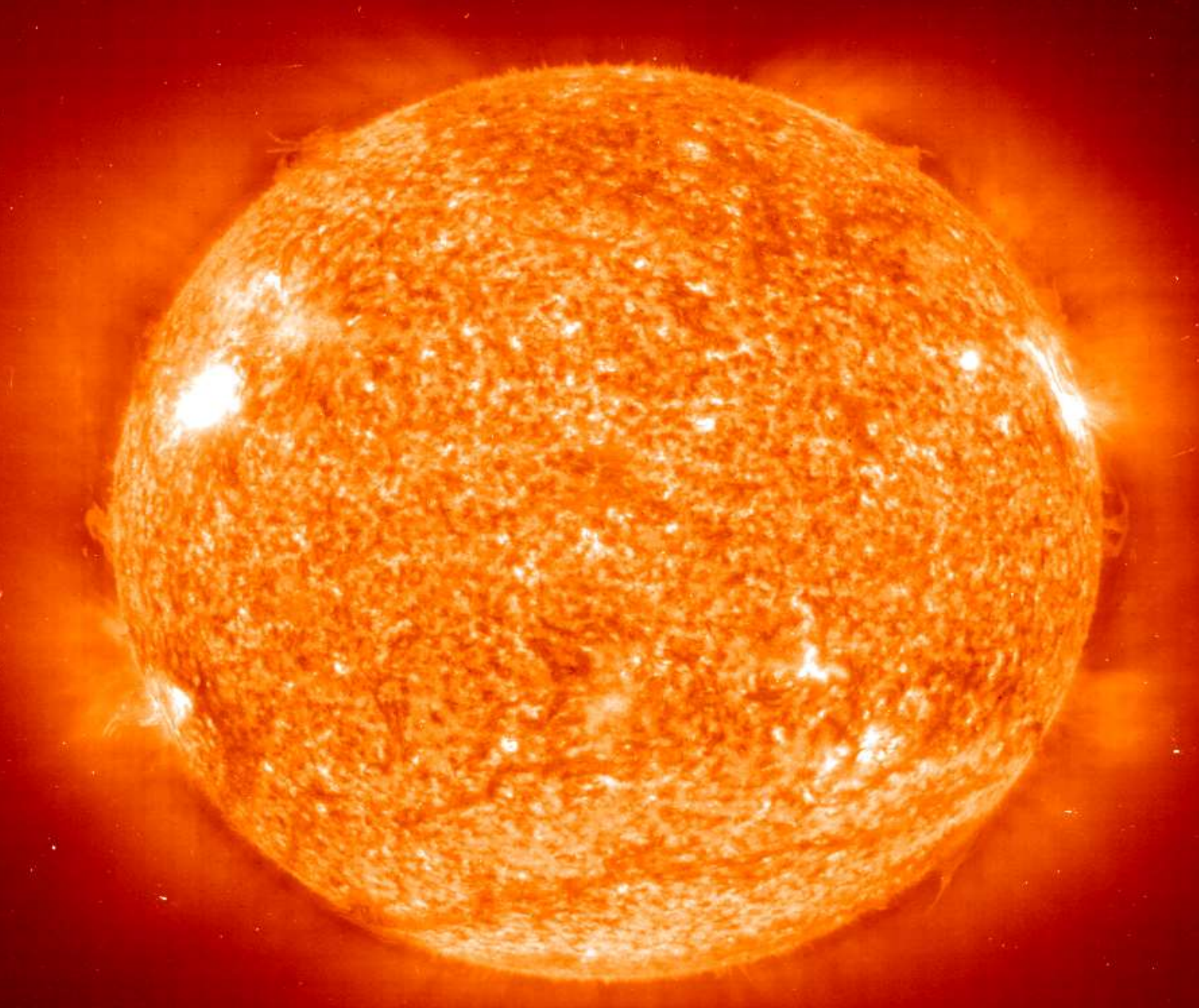




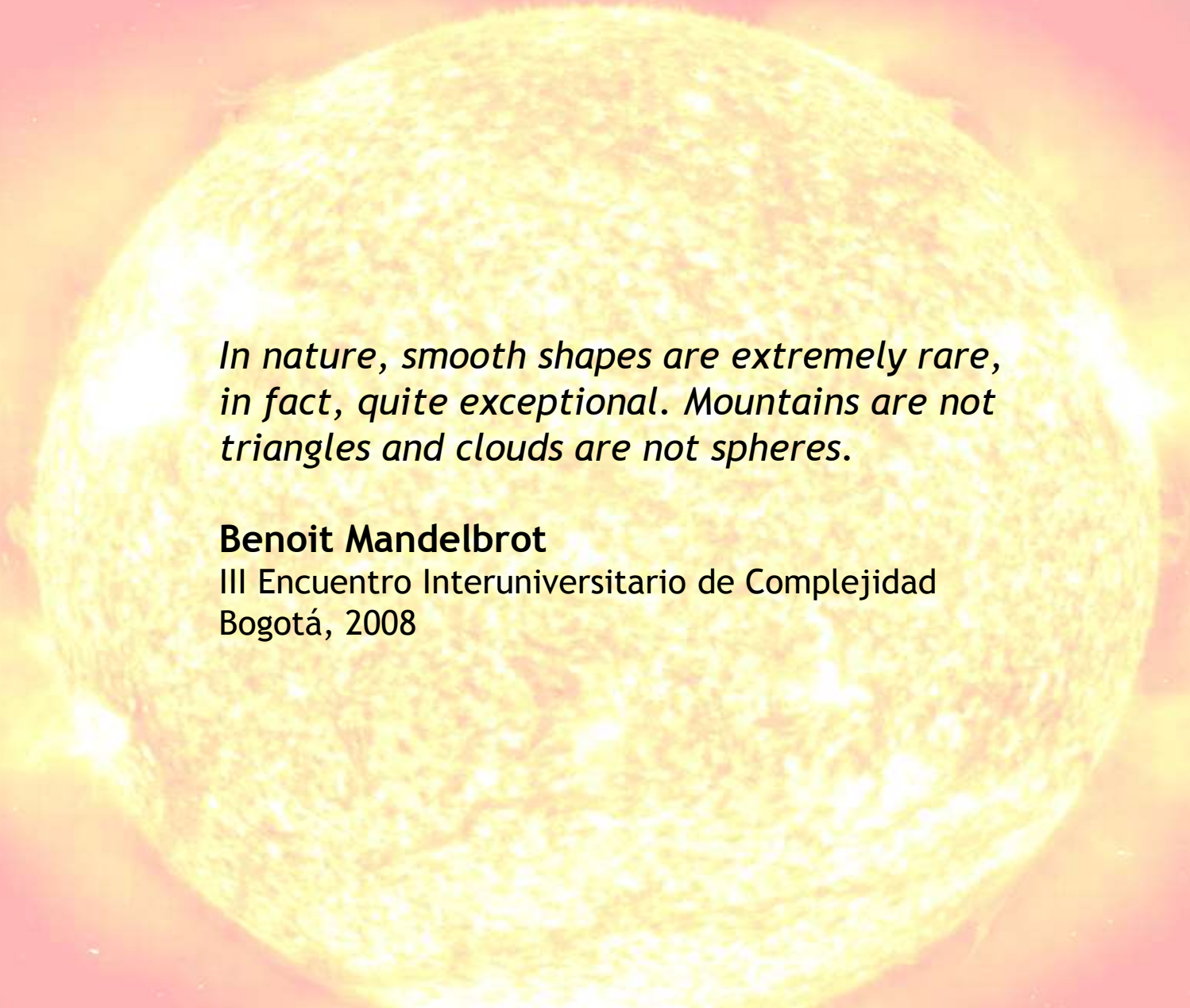








his unique geometrical insights



*In nature, smooth shapes are extremely rare,
in fact, quite exceptional. Mountains are not
triangles and clouds are not spheres.*

Benoit Mandelbrot

III Encuentro Interuniversitario de Complejidad
Bogotá, 2008

QUINTA PARTE

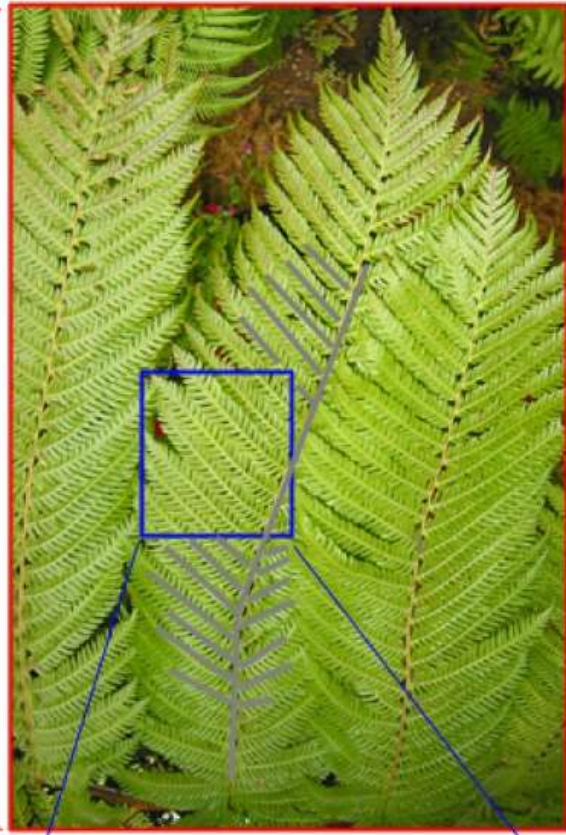
- ⦿ **He saw that infinite complexity
Could be described by simple rules
He used his giant brain
to turn the game around**





OTRA GEOMETRÍA

- ◉ Las formas principales no se construyen directamente con líneas rectas, círculos o triángulos.
- ◉ ... Son más bien un conjunto de procedimientos (algoritmos) para rotar, desplazar, aumentar, disminuir y/o distorsionar una forma original.



» Monroy
al F.J.C.

Infinite complexity, simple rules

TRANSFORMACIONES AFINES EN \mathbb{R}^2

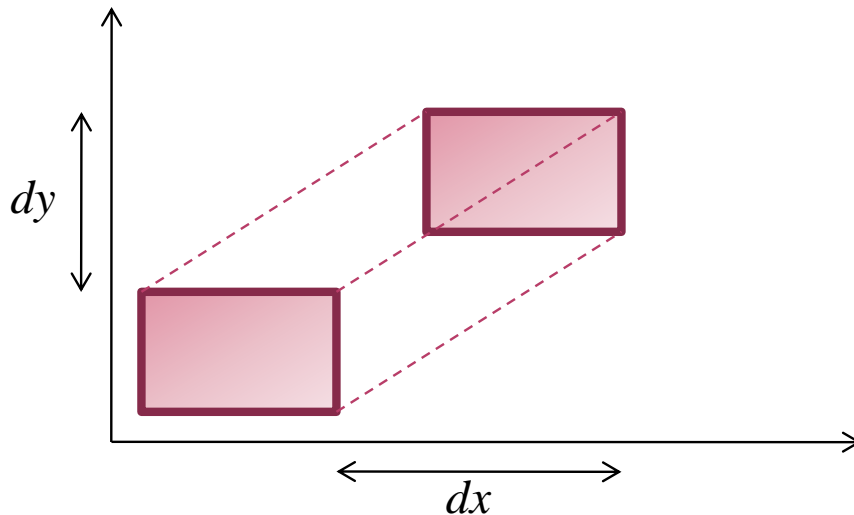
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$



Rotar
Escalar
Distorcionar

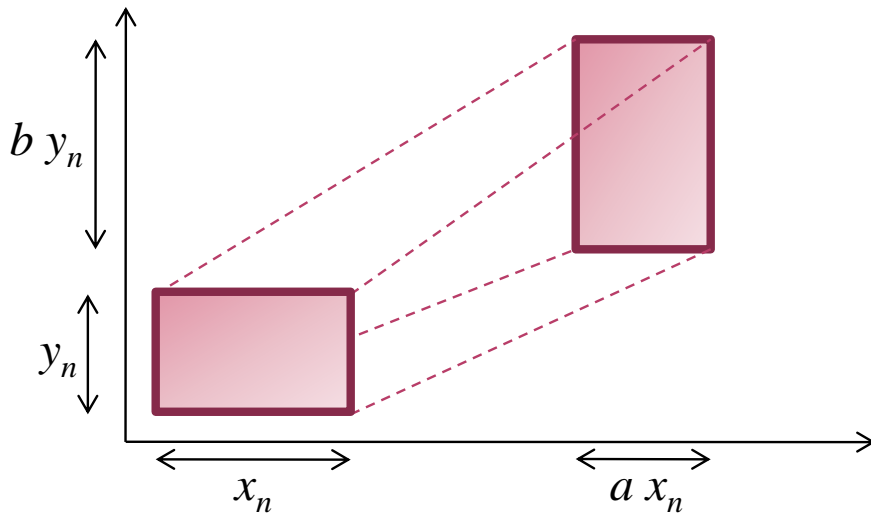


Desplazar



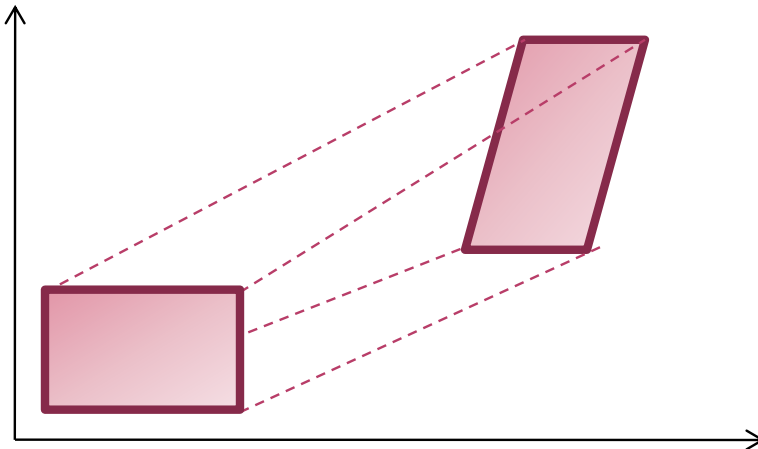
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Una traslación simple por (dx, dy)



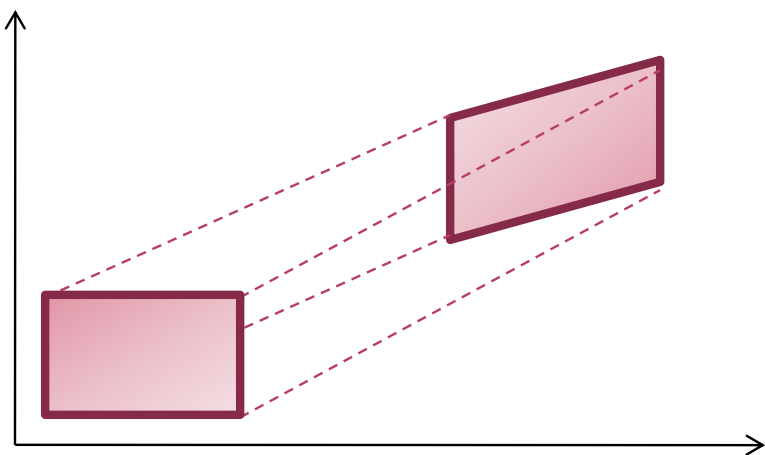
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Una re-escalización por (a,b) ,
con traslación -si $ab=1$,
preserva el área-



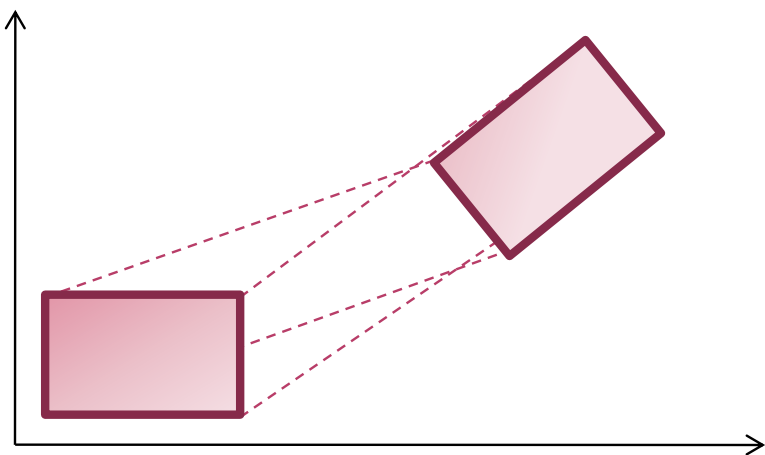
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Corrimiento paralelo a x



$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Corrimiento paralelo a y

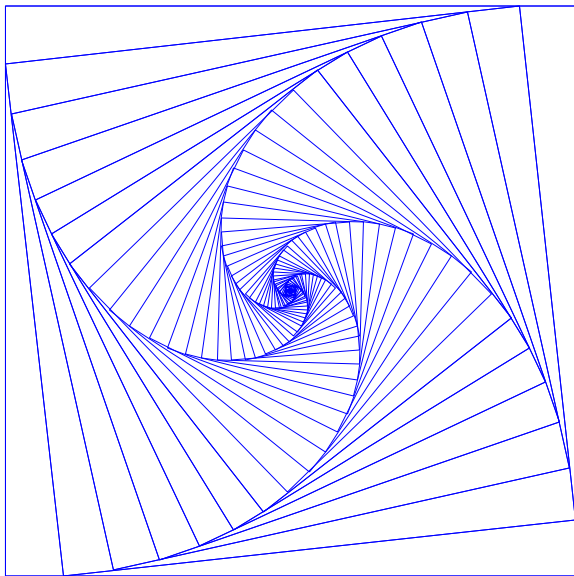


$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Rotación por un ángulo θ

PODEMOS ITERAR UNA TRANSFORMACIÓN AFÍN SIMPLE:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.0 \end{bmatrix}$$



Iterated
Function
System

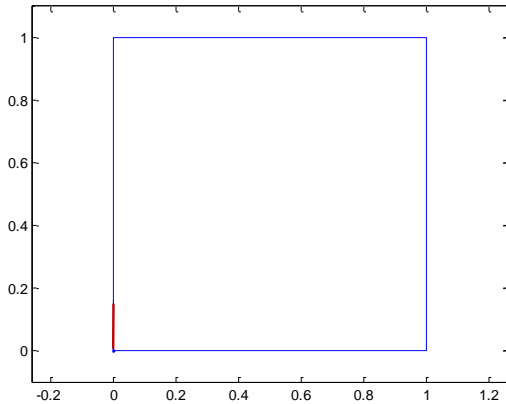
```
xy = [0 0; 1 0; 1 1; 0 1; 0 0]';  
A = [0.9 -0.1; 0.1 0.9];  
b = repmat([0.1; 0.0],1,5);  
for i = 1:100  
    plot(xy(1,:),xy(2,:)); hold on  
    xy = A*xy + b;  
end
```


CUATRO TRANSFORMACIONES AFINES INTERESANTES EN \mathbb{R}^2

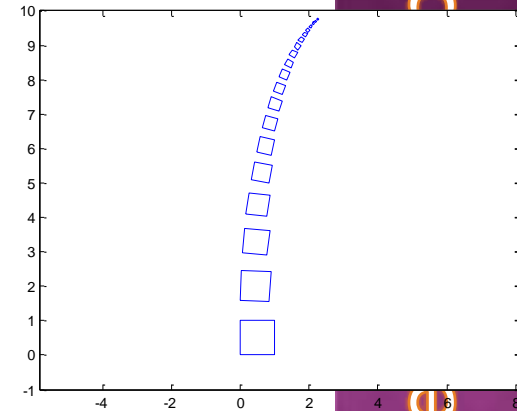
le rules

mple

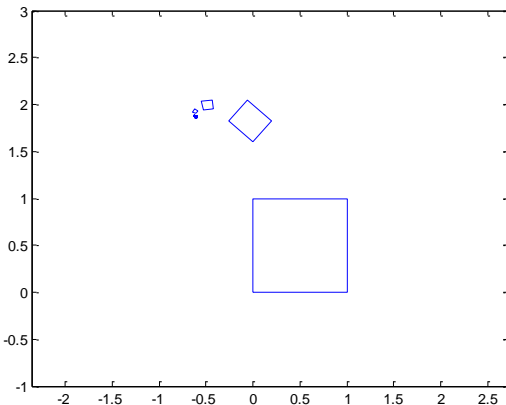
li



$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

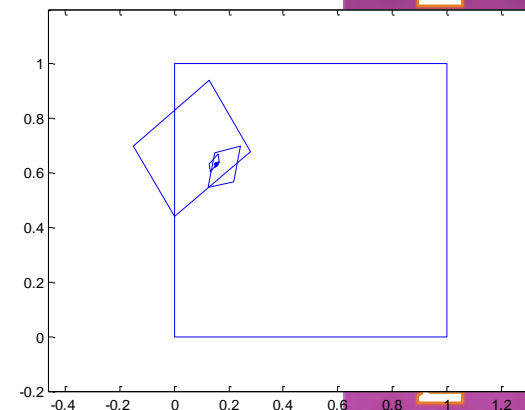


$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$



$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$



GEOMETRÍA DE LA NATURALEZA

Infinite complexity, simple rules



Empezando con $x = (0.5, 0.5)$:

plot(x)

con prob. 0.01, $x \leftarrow A_1 x + b_1$;

con prob. 0.85, $x \leftarrow A_2 x + b_2$;

con prob. 0.07, $x \leftarrow A_3 x + b_3$;

con prob. 0.07, $x \leftarrow A_4 x + b_4$;

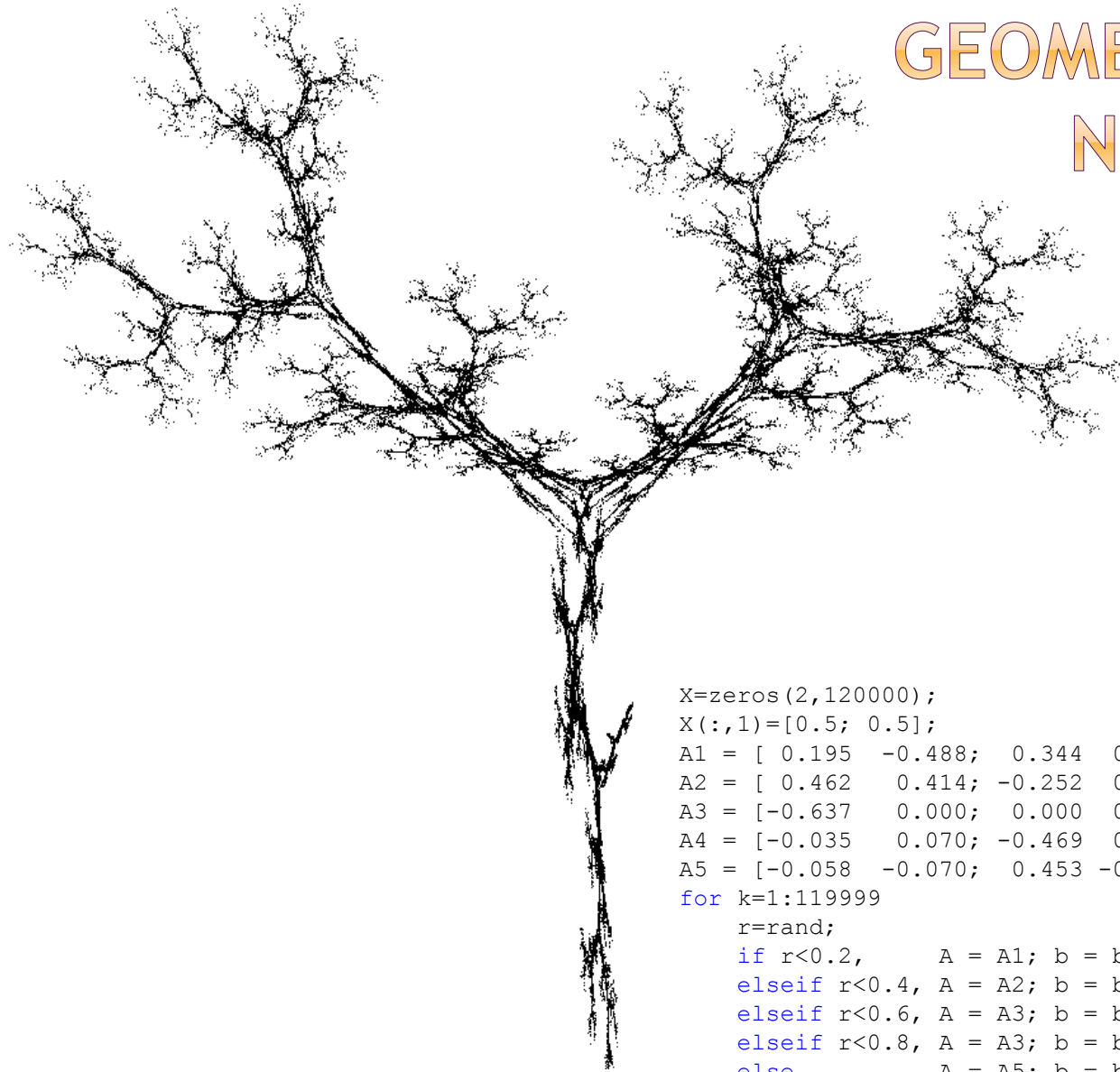
Repita hasta tener suficientes puntos

```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.00  0.00;  0.00  0.16]; b1 = [0.0; 0.00];
A2 = [ 0.85  0.04; -0.04  0.85]; b2 = [0.0; 1.60];
A3 = [ 0.20 -0.26;  0.23  0.22]; b3 = [0.0; 1.60];
A4 = [-0.15  0.28;  0.26  0.24]; b4 = [0.0; 0.44];
for k=1:119999
    r=rand;
    if      r<.01, A = A1; b = b1;
    elseif r<.86, A = A2; b = b2;
    elseif r<.93, A = A3; b = b3;
    else      A = A4; b = b4;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'g')
axis equal
```

Rock, conjunto de
Mandelbrot y π

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Universidad Distrital F.J.C.

GEOMETRÍA DE LA NATURALEZA

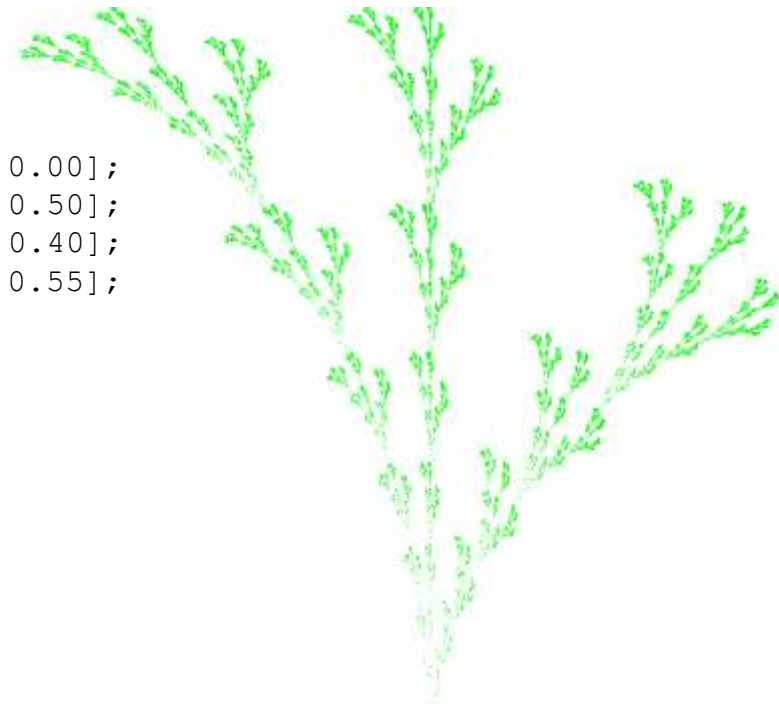


```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.195 -0.488; 0.344 0.443]; b1 = [0.4431; 0.2452];
A2 = [ 0.462 0.414; -0.252 0.361]; b2 = [0.2511; 0.5692];
A3 = [-0.637 0.000; 0.000 0.501]; b3 = [0.8562; 0.2512];
A4 = [-0.035 0.070; -0.469 0.022]; b4 = [0.4884; 0.5069];
A5 = [-0.058 -0.070; 0.453 -0.111]; b5 = [0.5976; 0.0969];
for k=1:119999
    r=rand;
    if r<0.2,      A = A1; b = b1;
    elseif r<0.4, A = A2; b = b2;
    elseif r<0.6, A = A3; b = b3;
    elseif r<0.8, A = A4; b = b4;
    else          A = A5; b = b5;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'k')
axis equal
```

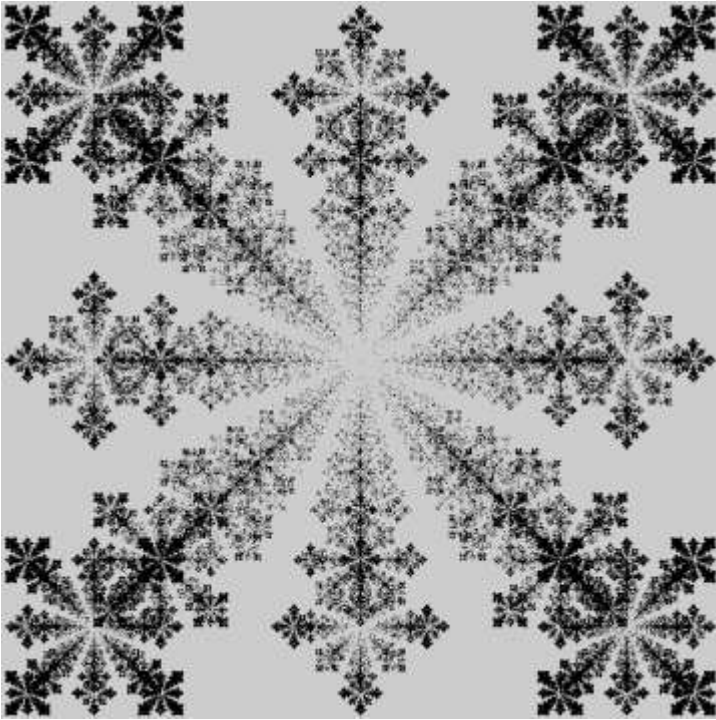


```
X=zeros(2 120000);
X(:,1)=[0 5; 0.5];
A1 = [ 0. 87  0.430;  0.430 -0.387]; b1 = [0.2560; 0.5220];
A2 = [ 0. 41 -0.091; -0.009 -0.322]; b2 = [0.4219; 0.5059];
A3 = [-0. 68  0.020; -0.113  0.015]; b3 = [0.4000; 0.4000];
for k=1:1 9999
    r=ran ;
    if r< 333,      A = A1; b = b1;
    elseif r<.666, A = A2; b = b2;
    else           A = A3; b = b3;
    end
    X(:,k 1) = A*X(:,k) + b;
end
scatter(X 1,:),X(2,:),2,'g')
```

```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.50  0.00;  0.00  0.75]; b1 = [0.25; 0.00];
A2 = [ 0.25 -0.20;  0.10  0.30]; b2 = [0.25; 0.50];
A3 = [ 0.25  0.20; -0.10  0.30]; b3 = [0.50; 0.40];
A4 = [ 0.20  0.00;  0.00  0.30]; b4 = [0.40; 0.55];
for k=1:119999
    r=rand;
    if r<0.25,      A = A1; b = b1;
    elseif r<0.50, A = A2; b = b2;
    elseif r<0.75, A = A3; b = b3;
    else           A = A4; b = b4;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'g')
axis equal
```



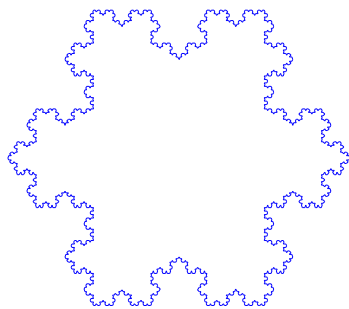
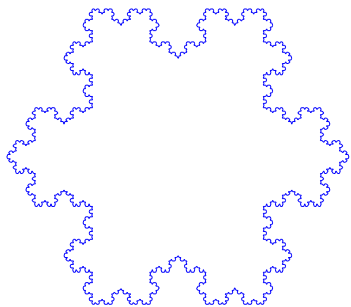
NO SÓLO PLANTAS



```
X=zeros(2,250000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.75  0.00;  0.00  0.75]; b1 = [0.125; 0.125];
A2 = [ 0.50 -0.50;  0.50  0.50]; b2 = [0.500; 0.000];
A3 = [ 0.25  0.00;  0.00  0.25]; b3 = [0.000; 0.750];
A4 = [ 0.25  0.00;  0.00  0.25]; b4 = [0.750; 0.750];
A5 = [ 0.25  0.00;  0.00  0.25]; b5 = [0.000; 0.000];
A6 = [ 0.25  0.00;  0.00  0.25]; b6 = [0.750; 0.000];
for k=1:249999
    r=rand;
    if r<0.1667,      A = A1; b = b1;
    elseif r<0.3333, A = A2; b = b2;
    elseif r<0.5,    A = A3; b = b3;
    elseif r<0.6667, A = A4; b = b4;
    elseif r<0.8333, A = A5; b = b5;
    else             A = A6; b = b6;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'k')
axis equal
```

Infinite complexity, simple rules

Simetría y crecimiento



Rock, conjunto de
Mandelbrot y π

Marco Aurelio Alzate Monroy
Universidad Distrital F.J.C.

Infinite complexity, simple rules

EL COMIENZO DE LA FASCINACIÓN POPULAR CON LOS FRACTALES



- ◉ Benoit Mandelbrot, *“How long is the coast of Britain?”*, Science, Vol. 156, No. 3775. (May 5, 1967), pp. 636-638
- ◉ *“Why this text came to be written? It was intended to be a “Trojan” horse allowing a bit of mathematical esoterica to “infiltrate” surreptitiously hence near-painlessly, the investigation of the messiness of raw nature. Today, it means that everyone knows how to answer the question raised in this paper's title. And the notion of fractal dimension is very widely known and used.”*

¿CUÁNTO MIDE LA COSTA DE GRAN BRETAÑA?



Unidad = 200 km,
Longitud = 2400 km



Unidad = 100 km,
Longitud = 2800 km



Unidad = 50 km,
Longitud = 3400 km

Could be described by simple rules

DIMENSIONES

Informalmente, la dimensión de un objeto es el mínimo número de coordenadas necesarias para especificar cada punto en él.



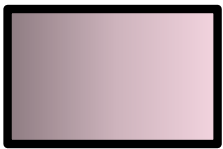
◉ Dimensión 0 $\{0\}$

Medida : Número de puntos



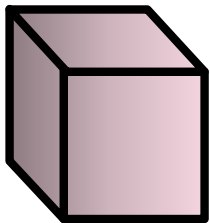
◉ Dimensión 1 (x)

Medida : Longitud



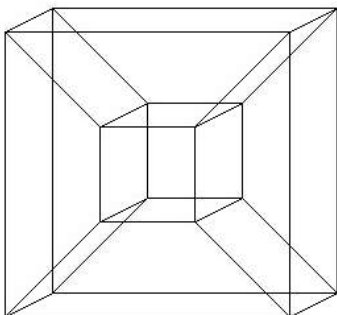
◉ Dimensión 2 (x,y)

Medida : Area



◉ Dimensión 3 (x,y,z)

Medida : Volumen

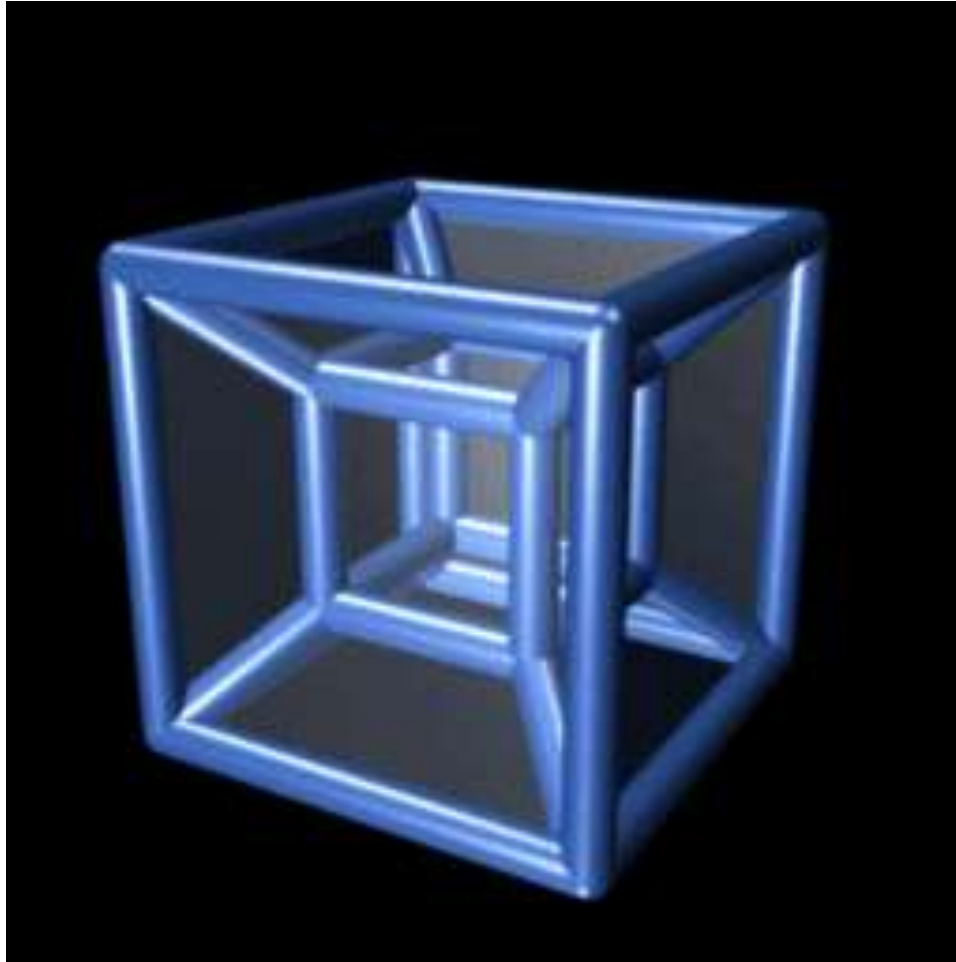


◉ Dimensión 4 (w,x,y,z)

Tesseract

(sombra de un hipercubo 4-dimensional)

TESSERACT

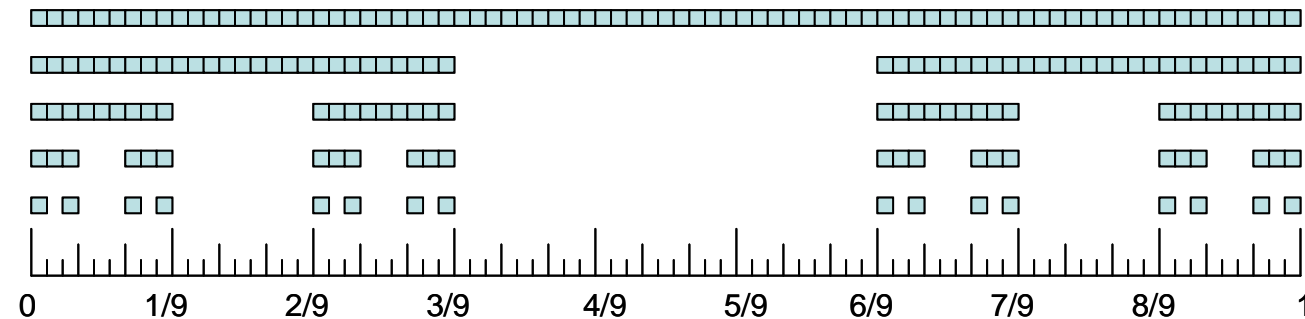


Rock, conjunto de
Mandelbrot y π

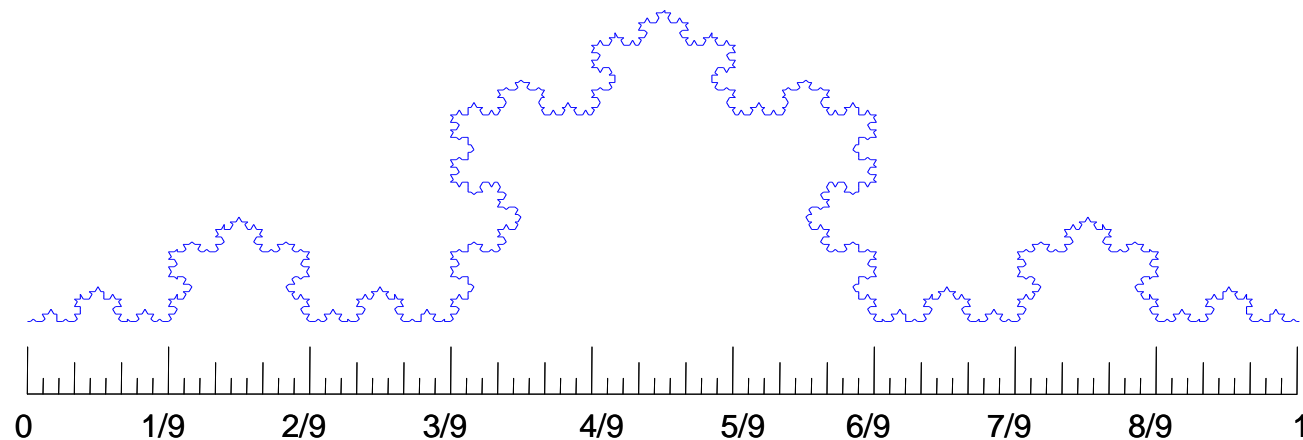
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his unique geometrical insights

FRACTALES AUTOSEMEJANTES

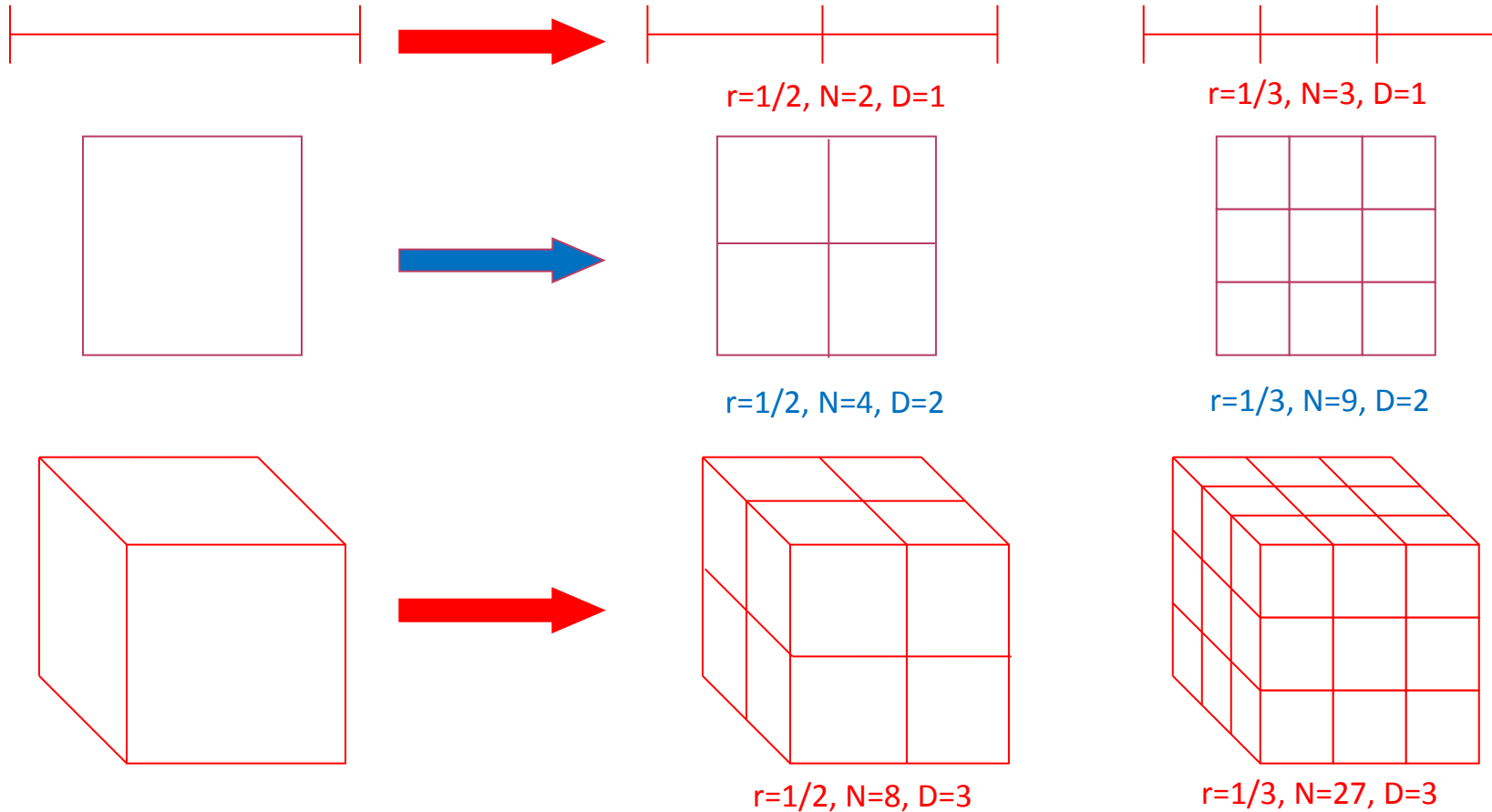


La dimensión del conjunto de Cantor no puede ser 1 porque su longitud es **cero**, pero debe ser mayor que cero porque tiene un número **infinito** no contable de puntos



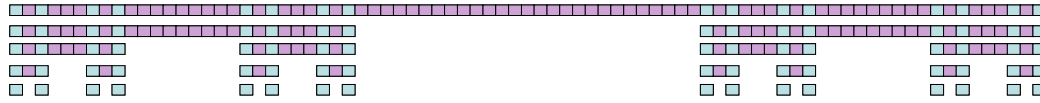
La dimensión de la curva de Koch no puede ser 2 porque su área es **cero**, pero debe ser mayor que uno porque tiene un número **infinito** no contable de puntos

(UN) CONCEPTO DE DIMENSIÓN (MEDIANTE AUTOSEMEJANZA)

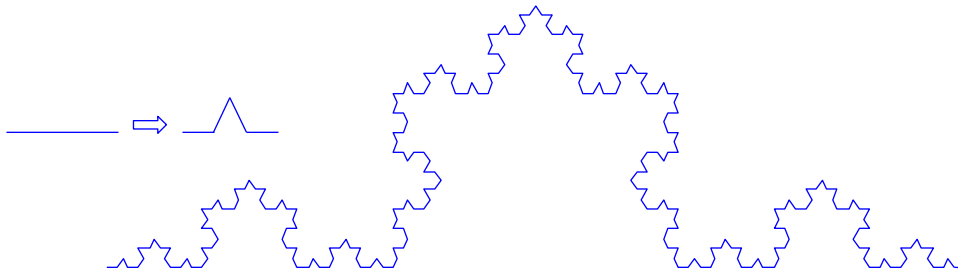


$$N = r^{-D} \Rightarrow D = \log(N) / \log(1/r)$$

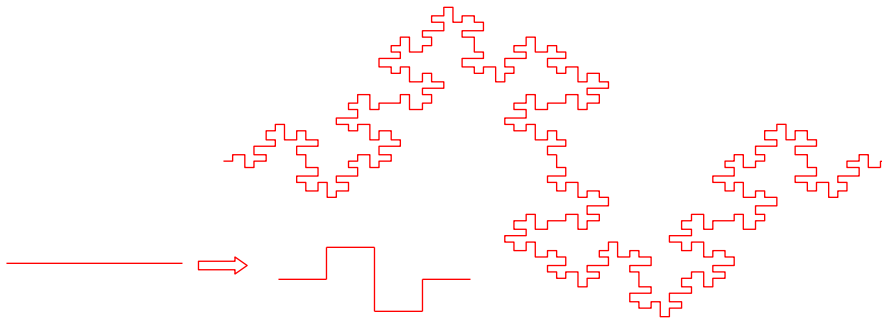
DIMENSIÓN DE FIGURAS AUTOSEMEJANTES



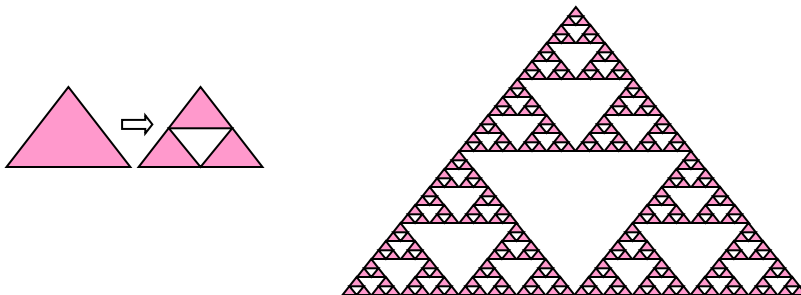
$$\begin{aligned} N &= 2, \\ r &= 1/3, \\ D &= \log(2)/\log(3) = 0.63 \end{aligned}$$



$$\begin{aligned} N &= 4, \\ r &= 1/3, \\ D &= \log(4)/\log(3) = 1.26 \end{aligned}$$



$$\begin{aligned} N &= 8, \\ r &= 1/4, \\ D &= \log(8)/\log(4) = 1.5 \end{aligned}$$



$$\begin{aligned} N &= 3, \\ r &= 1/2, \\ D &= \log(3)/\log(2) = 1.58 \end{aligned}$$

to turn the game around

ALGUNAS DIMENSIONES

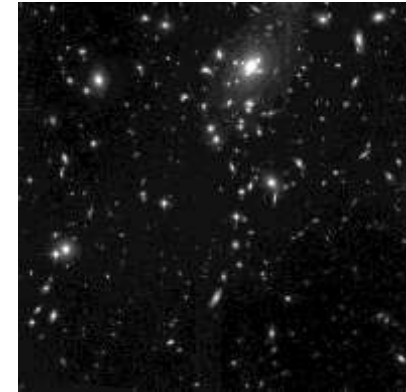
costa de Gran Bretaña : 1.25



Costa de
Noruega : 1.52



Agregación por difusión
limitada : 1.7



Grupos de galaxias : 2



Coliflor : 2.33



Papel arrugado : 2.5



Lámpara de
Lichtenberg : 2.5



Broccoli : 2.66



Superficie del
cerebro humano : 2.79



Alvéolos pulmonares : 2.97

**¿Cómo pueden medirse
estas dimensiones si no se
trata de objetos
autosemejantes?**

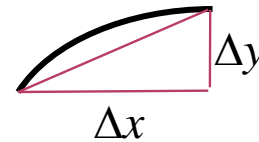
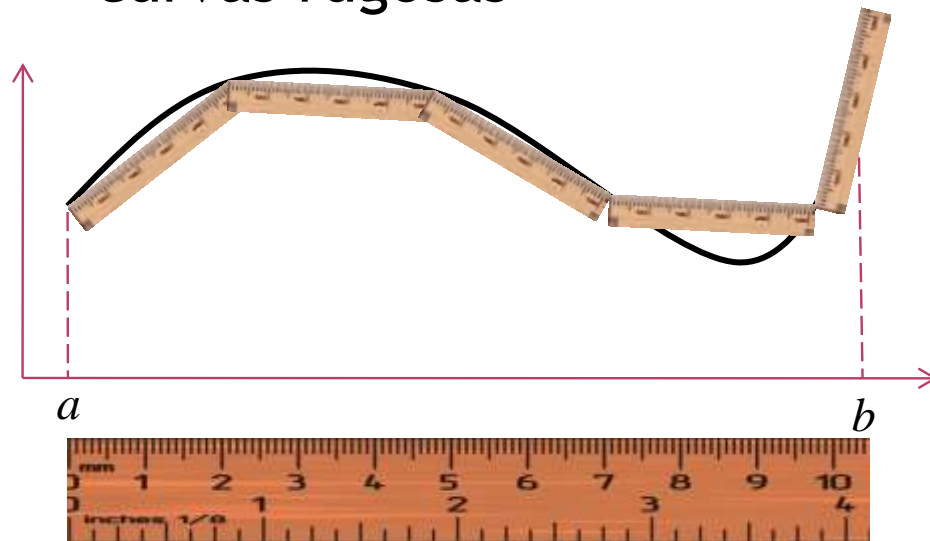
Rock, conjunto de
Mandelbrot y π

Marco Aurelio Alzate Monroy
Universidad Distrital F.J.C.

Face the demons down

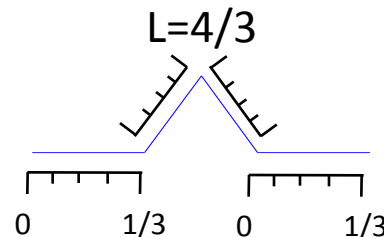
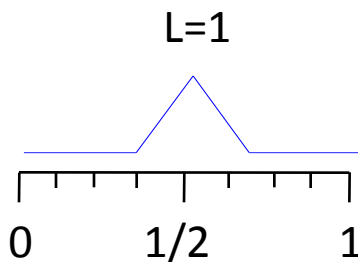
MUCHAS DEFINICIONES DE DIMENSION

- Basicamente: Ignore las irregularidades menores que ε y después estudie qué pasa cuando $\varepsilon \rightarrow 0$.
- Esto fue lo que hicimos antes con curvas suaves y con curvas rugosas



$$L = \sum_i \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i$$

$$\Rightarrow L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



LA DIMENSION DE HAUSDORFF

COBERTURA- δ DE UN CONJUNTO F

- Si U es un subconjunto no vacío del espacio euclidiano n -dimensional \mathbb{R}^n , ...
- ... el diámetro de U se define como

$$|U| = \sup \{ |x - y| : x, y \in U \}$$

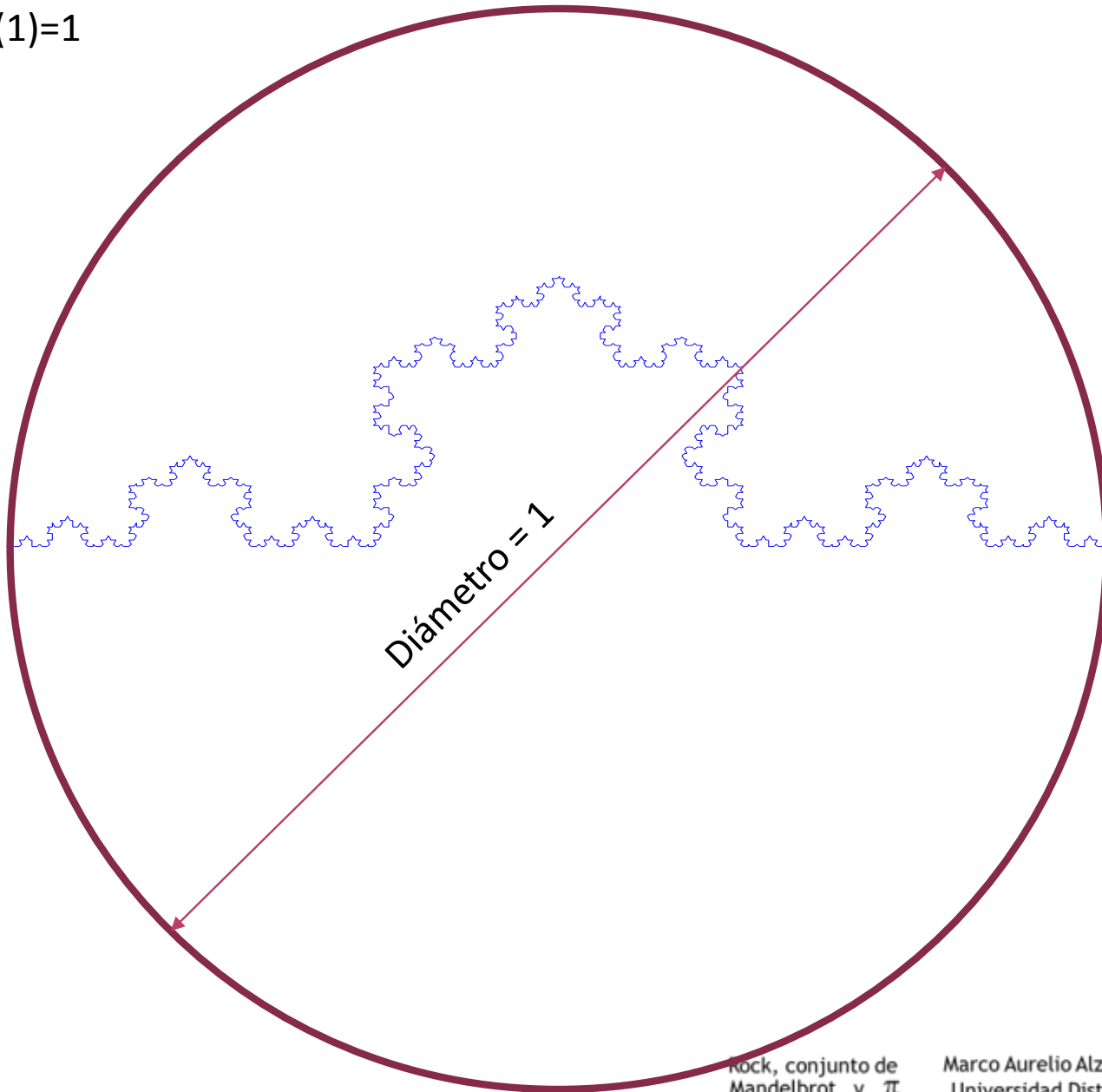
- Si $\{U_i\}$ es una colección contable de conjuntos de diámetro no mayor a δ que cubre a F , esto es,

$$F \subset \bigcup_{i=1}^{\infty} U_i$$

- ... con $0 \leq |U_i| \leq \delta$ para cada i , ...
- ... decimos que $\{U_i\}$ es una cobertura- δ de F

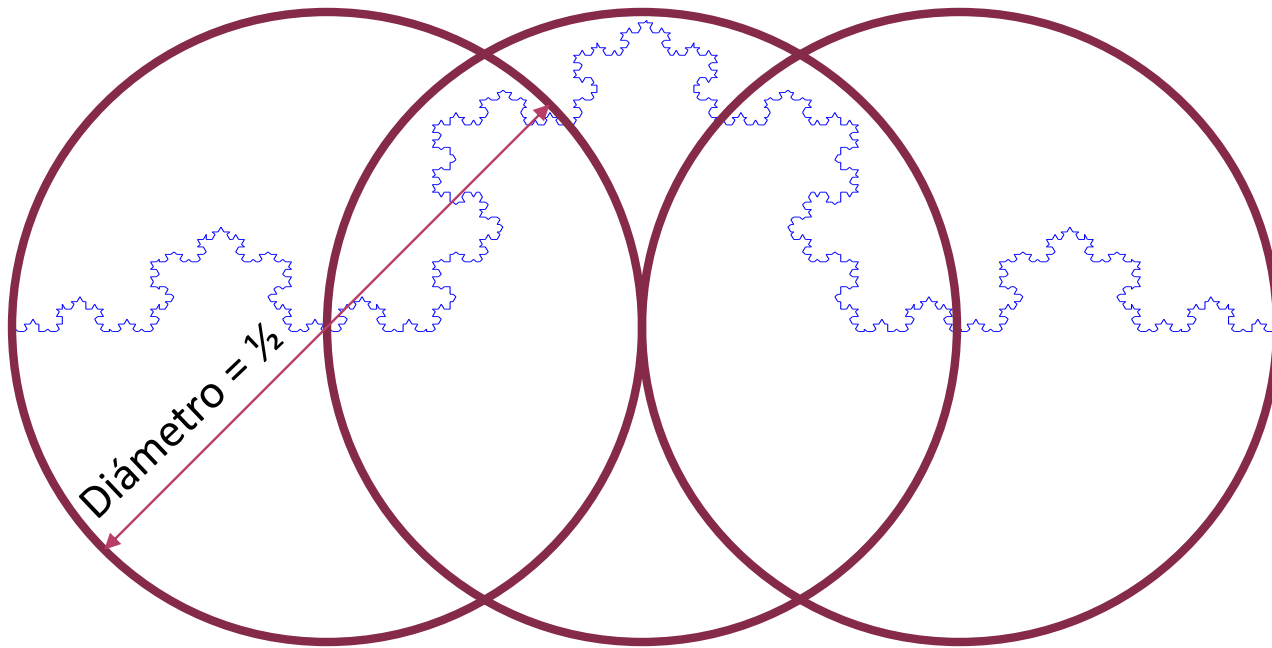
cobertura-1 de la curva de Koch

$$N(1)=1$$



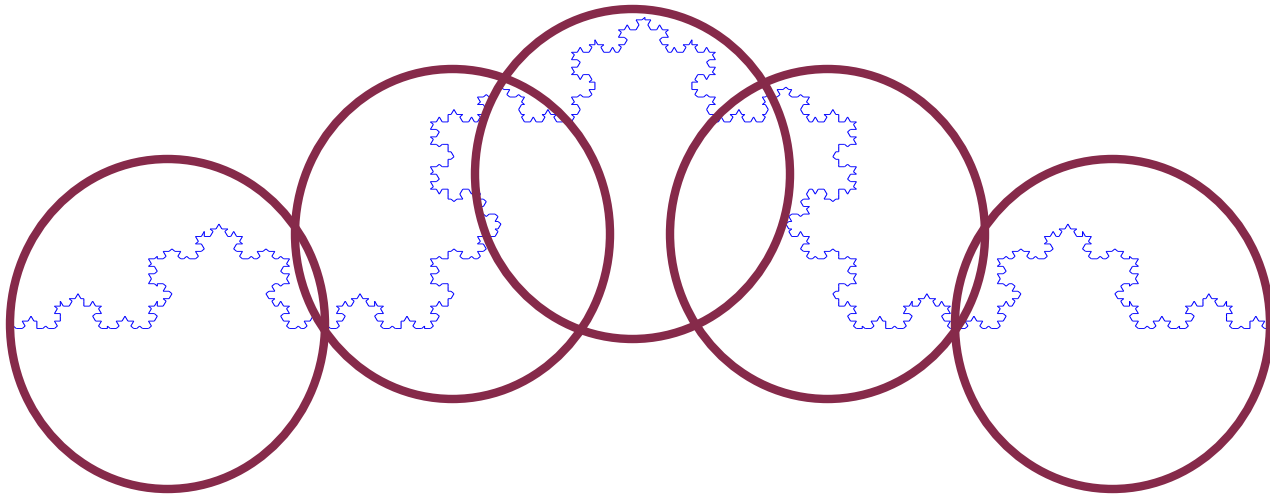
cobertura- $\frac{1}{2}$ de la curva de Koch

$$N(\frac{1}{2})=3$$



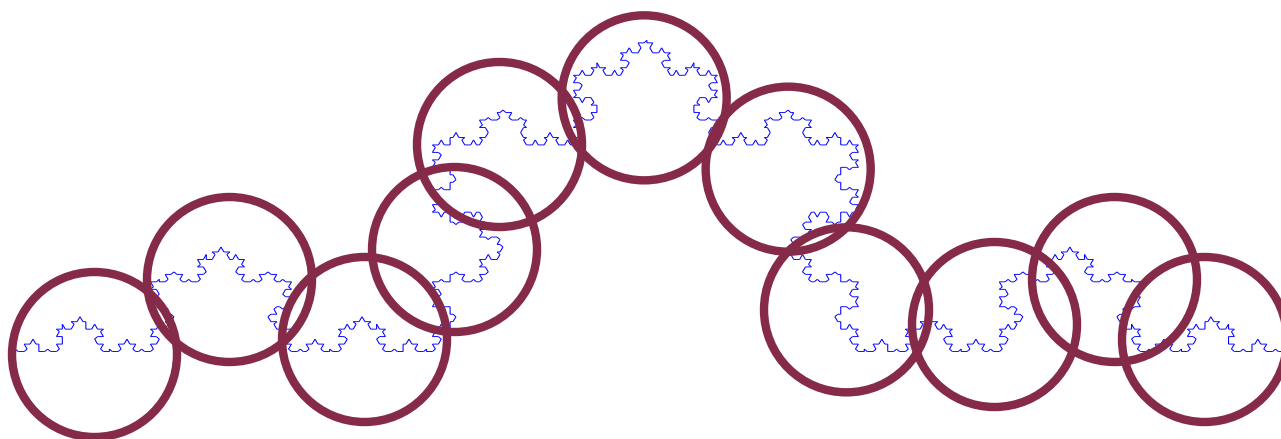
cobertura- $\frac{1}{4}$ de la curva de Koch

$$N(\frac{1}{4})=5$$



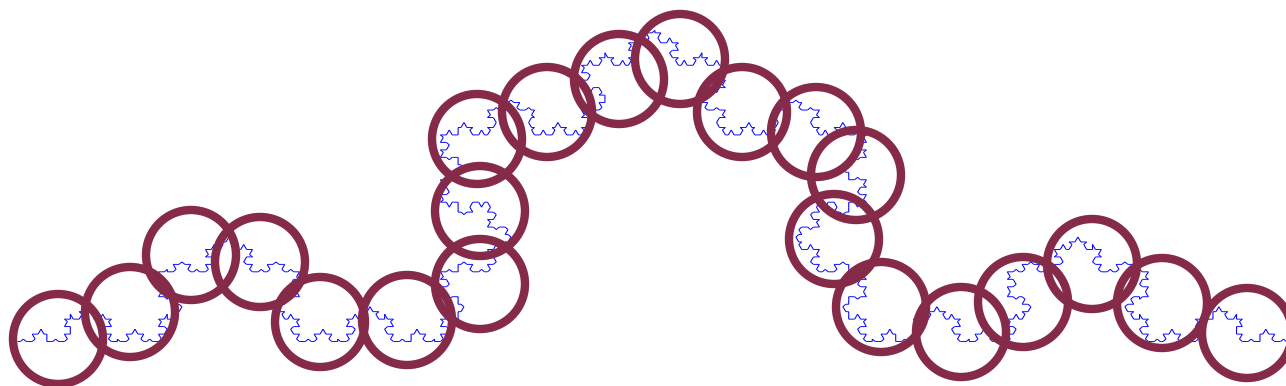
cobertura-1/7 de la curva de Koch

$$N(1/7)=11$$



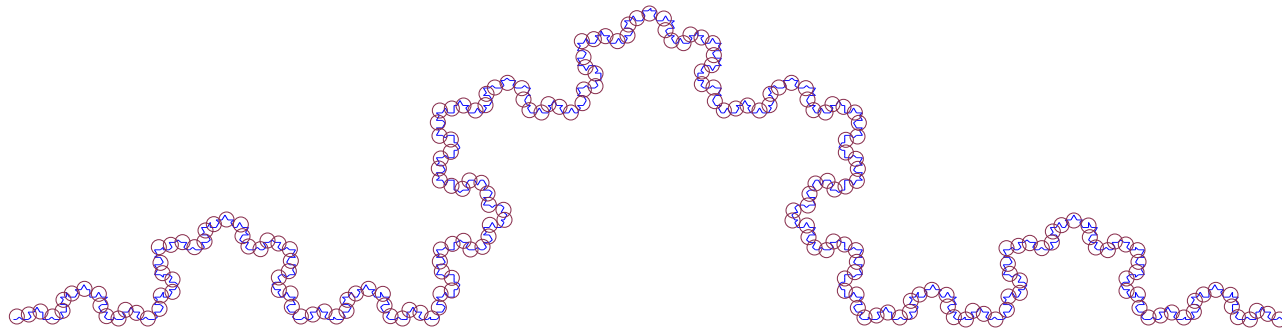
cobertura-1/12 de la curva de Koch

$$N(1/12)=22$$



cobertura- δ de la curva de Koch, $\delta = 1/88$

$$N(1/88)=244$$



MEDIDA DE HAUSDORFF

- Suponga que F es un subconjunto de \mathbb{R}^n y que s es un número no-negativo. Para cualquier $\delta > 0$ definimos

$$H_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

- Cuando δ se reduce, la clase de coberturas- δ de F permitidas se reduce, así que el ínfimo crece, por lo que se acerca a un límite cuando $\delta \rightarrow 0$:

$$H^s(F) = \lim_{\delta \rightarrow 0} H_\delta^s(F)$$

Medida Hausdorff s -dimensional del conjunto F

$$\delta = 1/7, s = 1$$

$$\sum_{i=1}^{12} \delta^1 = 12\delta \quad \checkmark$$

$$\sum_{i=1}^{11} \delta^1 = 11\delta \quad \checkmark$$

$$\sum_{i=1}^{10} \delta^1 = 10\delta \quad \times$$

his unique geometrical insights

$$H_{1/7}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ es una cobertura-(1/7) de } F \right\} = \frac{11}{7}$$

$$H_1^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (1)-cover of } F \right\} = 1 \cdot 1^1 = 1$$

$$H_1^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (1)-cover of } F \right\} = 1 \cdot 1^2 = 1$$

$$H_{0.5}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (0.5)-cover of } F \right\} = 3 \cdot 0.5^1 = 1.5$$

$$H_{0.5}^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (0.5)-cover of } F \right\} = 3 \cdot 0.5^2 = 0.75$$

$$H_{0.25}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (0.25)-cover of } F \right\} = 5 \cdot 0.25^1 = 1.25$$

$$H_{0.25}^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (0.25)-cover of } F \right\} = 5 \cdot 0.25^2 = 0.3125$$

$$H_{0.143}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (0.143)-cover of } F \right\} = 11 \cdot 0.143^1 = 1.573$$

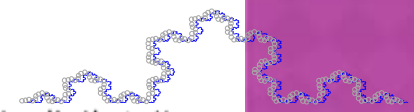
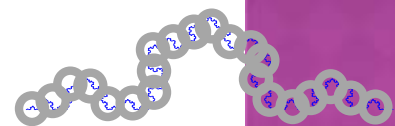
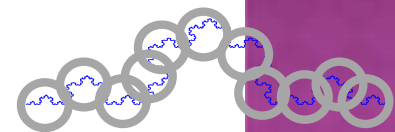
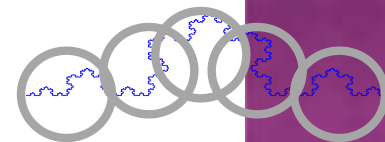
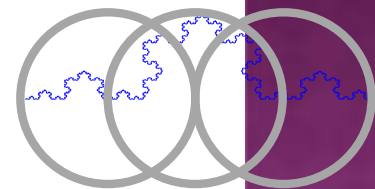
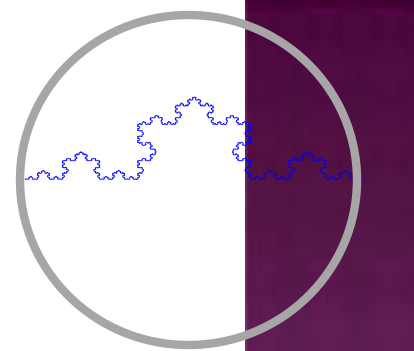
$$H_{0.143}^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (0.143)-cover of } F \right\} = 11 \cdot 0.143^2 = 0.2245$$

$$H_{0.083}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (0.083)-cover of } F \right\} = 22 \cdot 0.083^1 = 1.83$$

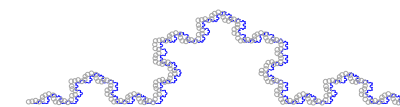
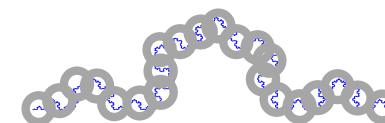
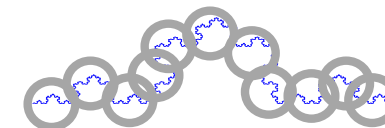
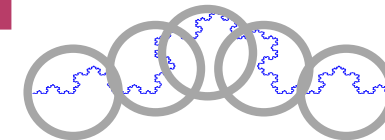
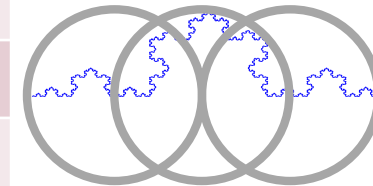
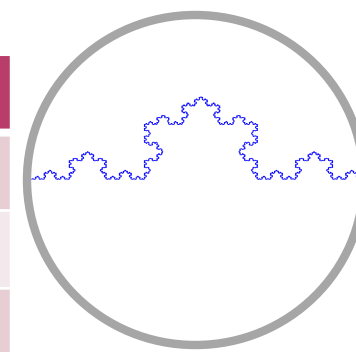
$$H_{0.083}^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (0.083)-cover of } F \right\} = 22 \cdot 0.083^2 = 0.153$$

$$H_{0.0114}^1(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a (0.0114)-cover of } F \right\} = 244 \cdot 0.0114^1 = 2.773$$

$$H_{0.0114}^2(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a (0.0114)-cover of } F \right\} = 244 \cdot 0.0114^2 = 0.0315$$



δ	$H_\delta^1(\text{Von Koch})$	$H_\delta^2(\text{Von Koch})$
1	1	1
1/2	3/2	3/4
1/4	5/4	5/16
1/7	11/7	11/49
1/12	22/12	22/144
1/88	244/88	244/7744
0	∞	0



La medida de Hausdorff

$$H^s(F) = \lim_{\delta \rightarrow 0} H_\delta^s(F)$$

Puede (y suele) ser 0 ó ∞



$H^0(F) = \text{Número de puntos}$

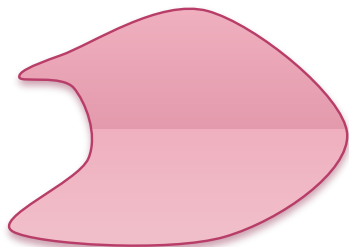
$$H^1(F) = 0$$



$$H^0(F) = \infty$$

$H^1(F) = \text{Longitud de la curva}$

$$H^2(F) = 0$$



$$H^1(F) = \infty$$

$H^2(F) \propto \text{Area de la superficie}$

$$H^3(F) = 0$$



$$H^2(F) = \infty$$

$H^3(F) \propto \text{Volumen del cuerpo}$

$$H^4(F) = 0$$

HACIA LA DIMENSIÓN DE HAUSDORFF

- ◉ Como

$$H_{\delta}^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ es una cobertura-}\delta \text{ de } F \right\}$$

- ◉ Para cualquier $\delta < 1$, $H_{\delta}^s(F)$ no crece con s . Consecuentemente

$$H^s(F) = \lim_{\delta \rightarrow 0} H_{\delta}^s(F)$$

no crece con s .

- ◉ Más aún, como

$$\sum_i |U_i|^t = \sum_i |U_i|^{t-s} |U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s$$

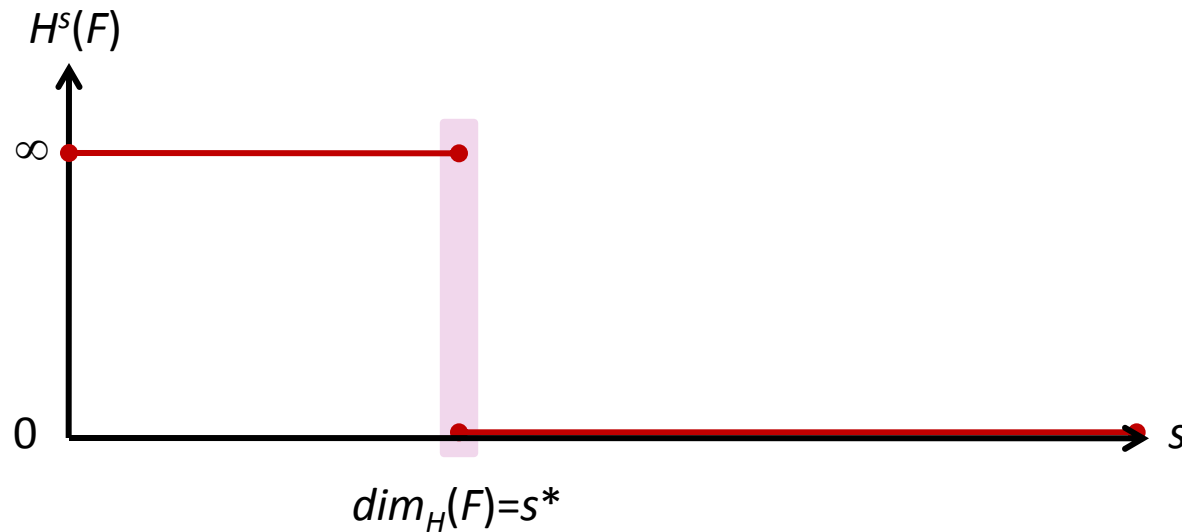
el ínfimo también debe obedecer esta relación

$$H_{\delta}^t(F) \leq \delta^{t-s} H_{\delta}^s(F)$$

- ◉ Por eso, en el límite,

$$\text{if } H^s(F) < \infty, H^t(F) = 0 \quad \forall \quad t > s$$

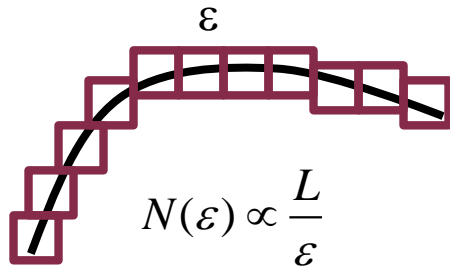
DIMENSIÓN HAUSDORFF



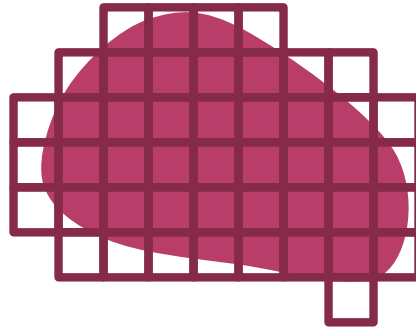
$$\dim_H(F) = \inf \left\{ s \geq 0 : H^s(F) = 0 \right\} = \sup \left\{ s \geq 0 : H^s(F) = \infty \right\}$$

EJEMPLO : DIMENSION-CAJA

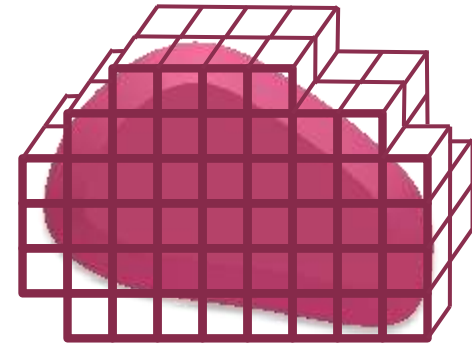
- Cubra el conjunto con cajas de tamaño ε



$$N(\varepsilon) \propto \frac{L}{\varepsilon}$$



$$N(\varepsilon) \propto \frac{A}{\varepsilon^2}$$



$$N(\varepsilon) \propto \frac{V}{\varepsilon^3}$$

- Así como con las líneas, superficies y objetos sólidos suaves, se puede esperar que la dimensión de un conjunto sea igual al exponente d en la siguiente ley de potencia

$$N(\varepsilon) \propto \frac{1}{\varepsilon^d}$$

EJEMPLO : DIMENSION-CAJA

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)}$$

- Encontrar la dimensión-caja del conjunto de Cantor



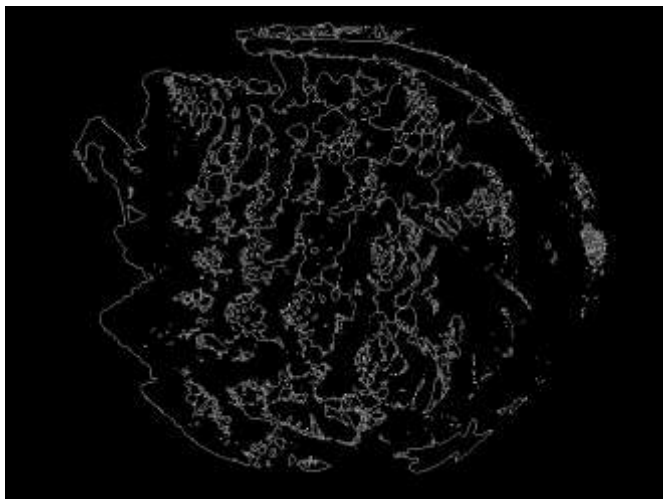
- Sea S_n el conjunto en el $n^{\text{ésimo}}$ paso.
- Para cada n , los 2^n intervalos of tamaño 3^{-n} de S_n cubren el conjunto de Cantor.
- Usando $\varepsilon = 3^{-n}$, necesitamos $N(\varepsilon) = 2^n$.
- Cuando $n \rightarrow \infty$, $\varepsilon \rightarrow 0$
- $$d = \lim_{n \rightarrow \infty} \frac{\log(2^n)}{\log(3^n)} = \frac{n \log(2)}{n \log(3)} = \frac{\log(2)}{\log(3)} = 0.631$$

DIMENSIÓN-CAJA DE UN FRACTAL BIDIMENSIONAL

```
% Carga y despliega la imagen original
p = imread('Fractal_Broccoli.jpg');
figure(1); imshow(p)
```

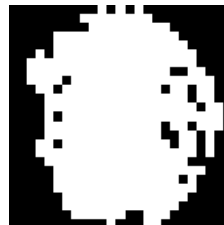


```
% detecta los bordes sobre la imagen B&N
p = edge(double(im2bw(p,graythresh(p))));
figure(2); imshow(p)
```



```
[Nx, Ny] = size(p); % Tamaño de la imagen
nBloques = [25 50 75 100 125 150]'; % Número de bloques
NB = length(nBloques);
tabla = zeros(NB,1);
for fg = 1:NB
    tamanoBloque_x = floor(Nx./nBloques(fg));
    tamanoBloque_y = floor(Ny./nBloques(fg));
    ocupado = zeros(nBloques(fg),nBloques(fg));
    for i = 1:nBloques(fg)
        xi = (i-1)*tamanoBloque_x + 1;
        xf = i*tamanoBloque_x;
        for j = 1:nBloques(fg)
            yi = (j-1)*tamanoBloque_y + 1;
            yf = j*tamanoBloque_y;
            bloque = p(xi:xf,yi:yf);
            ocupado(i,j) = any(bloque(:));
        end
    end
    tabla(fg) = nnz(ocupado);
figure(3); subplot(2,3,fg); imshow(ocupado);
end
```

eps = 0.04, N(eps) = 369



eps = 0.02, N(eps) = 1114



eps = 0.0133, N(eps) = 2055



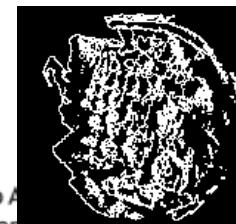
eps = 0.01, N(eps) = 3091



eps = 0.008, N(eps) = 4082



eps = 0.0067, N(eps) = 5430

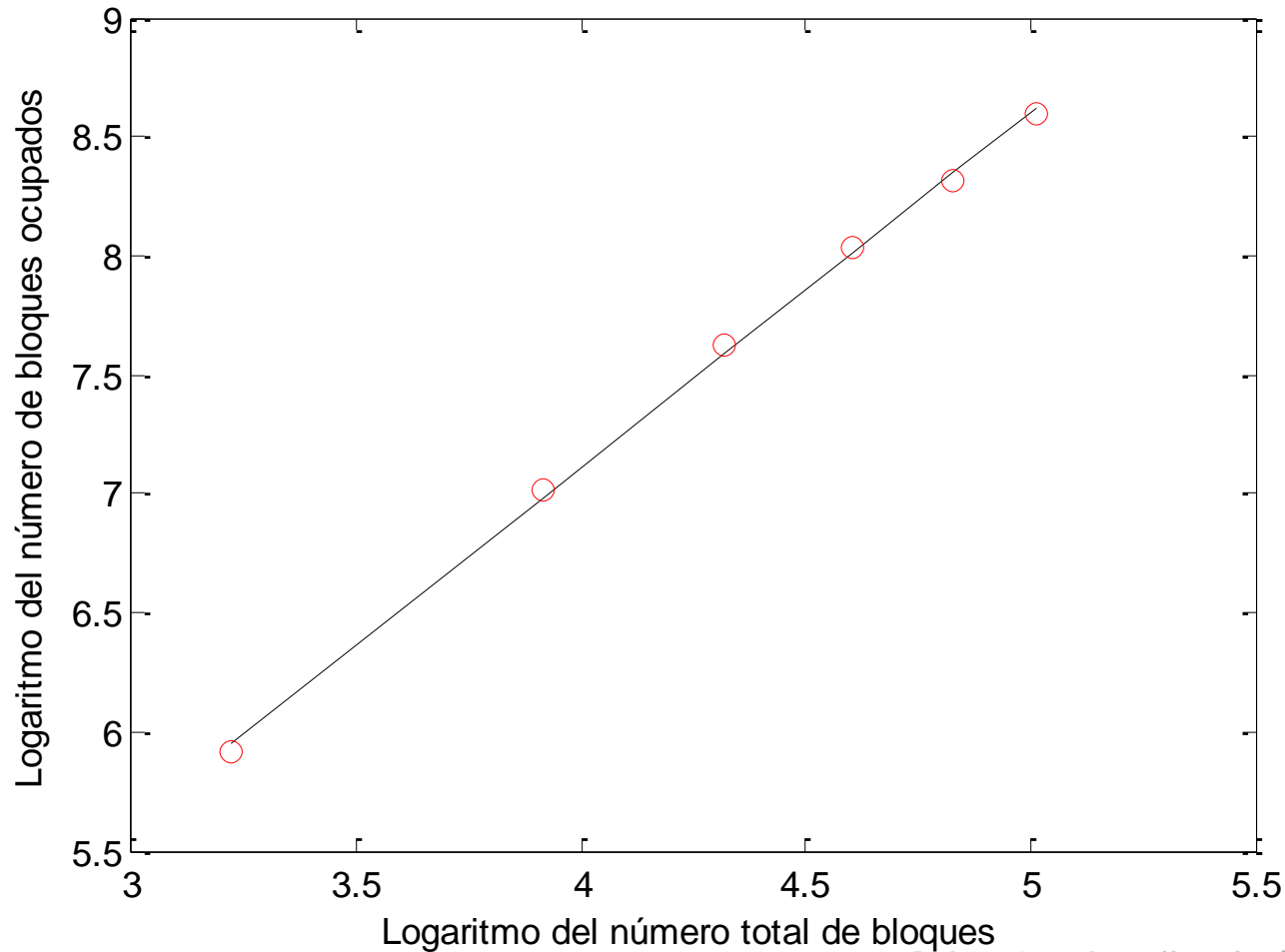


his unique geometrical insights

DIMENSIÓN-CAJA DE UN FRACTAL BIDIMENSIONAL

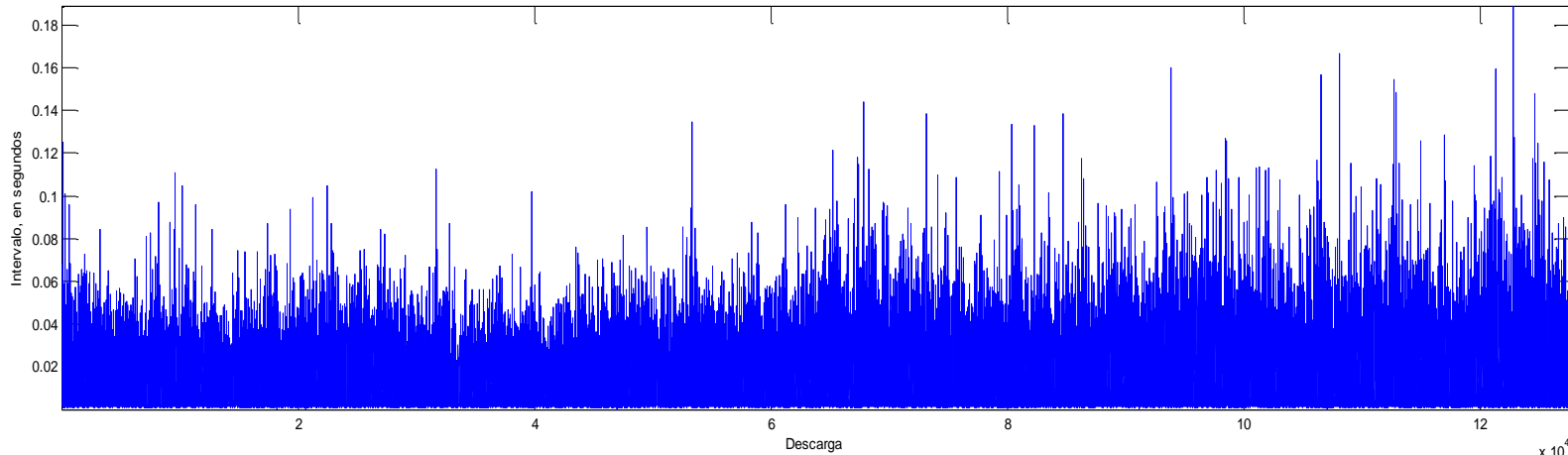
```
x = log(nBloques);  
y = log(tabla);  
p = polyfit(x,y,1);  
y1 = polyval(p,x);  
figure(4); plot(x,y1,'k-',x,y,'ro')  
Dimension_de_Hausdorff = p(1)
```

Dimension_de_Hausdorff = 1.4921



FRACTALES ALEATORIOS

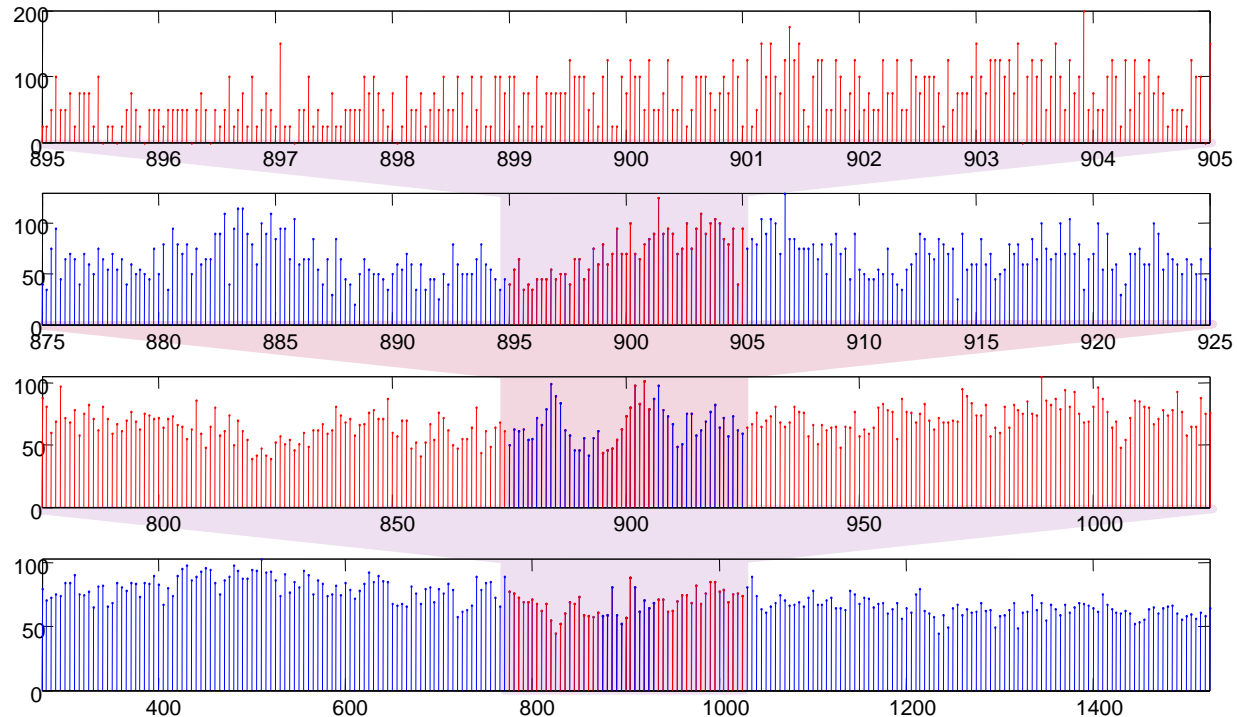
Intervalos entre descargas de una neurona que responde a un estímulo auditivo



Intervalos entre descargas de una neurona que responde a un estímulo auditivo

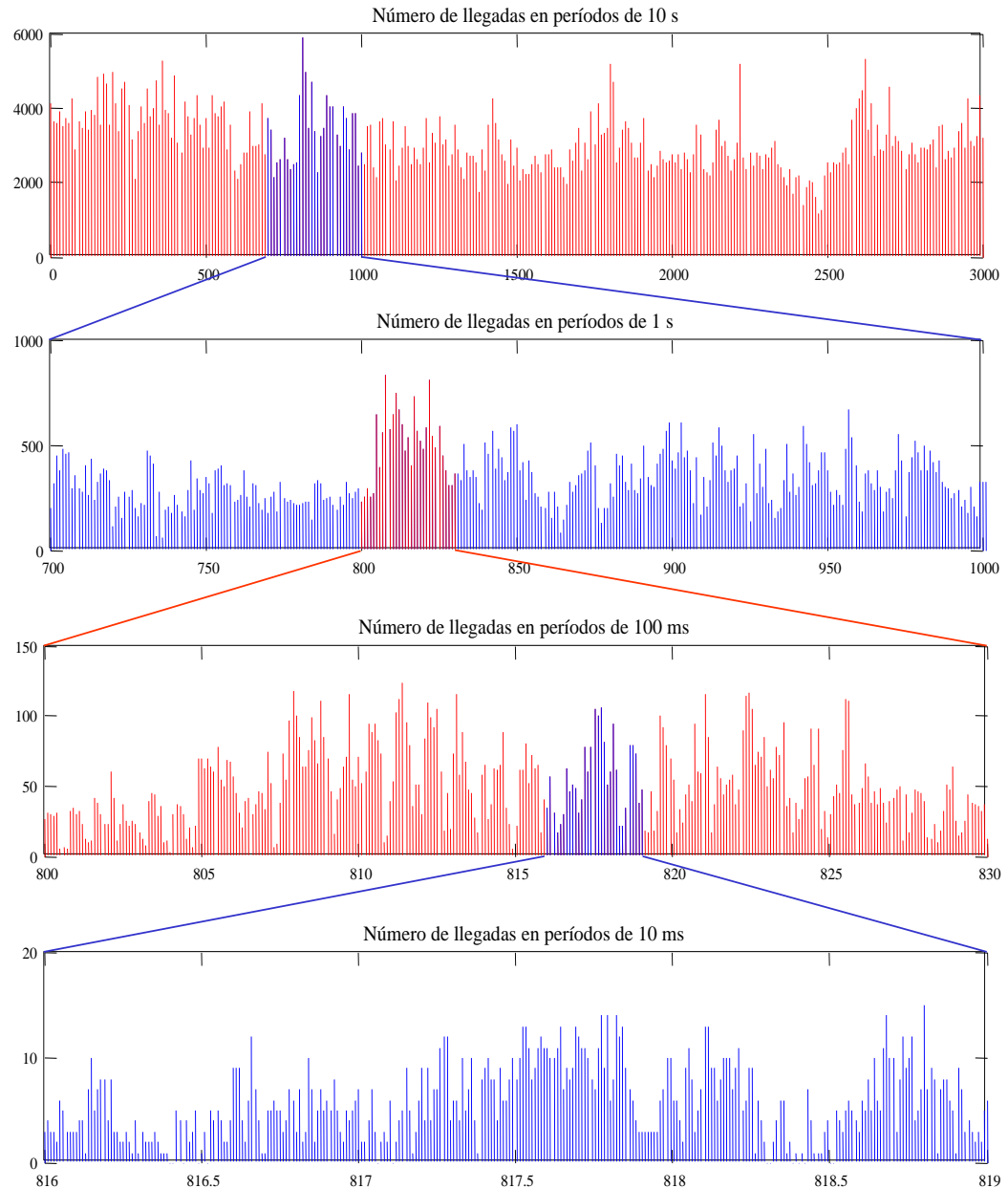
250 muestras del
número promedio
de disparos por
segundo, en
períodos de

- 40 ms
- 200 ms
- 1000 ms
- 5000 ms



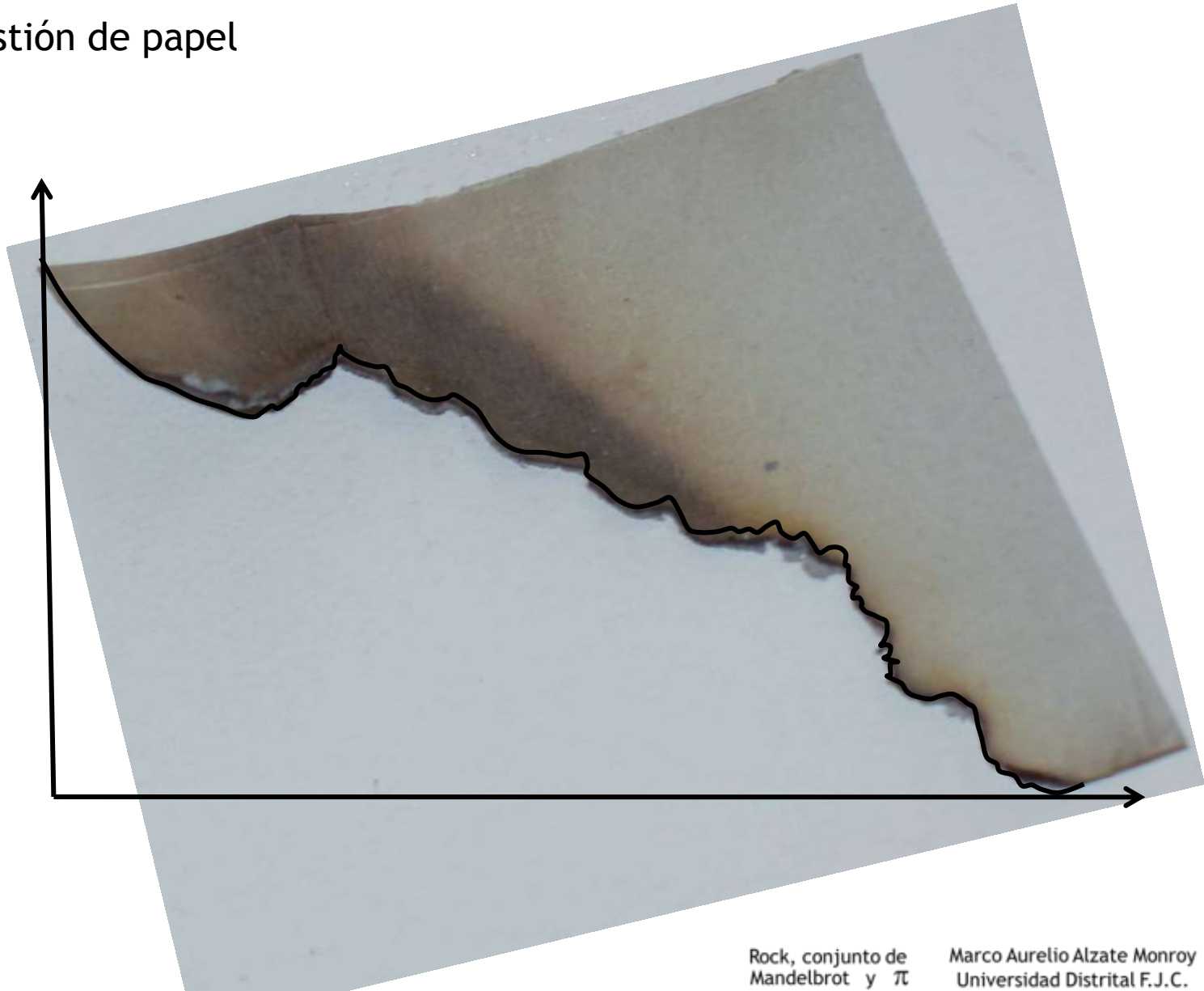
FRACTALES ALEATORIOS

Tráfico de datos en una
Red de comunicaciones



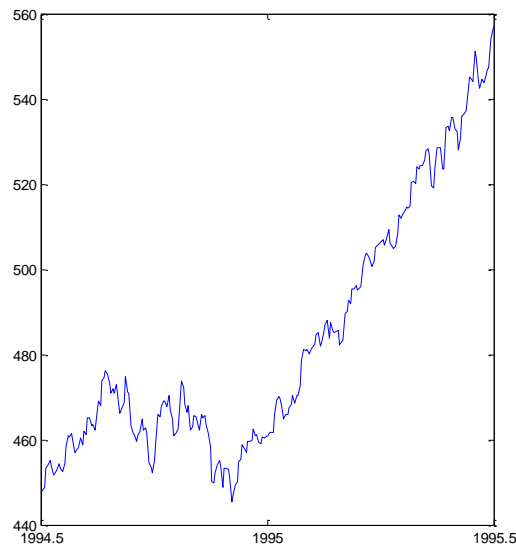
FRACTALES ALEATORIOS

Combustión de papel

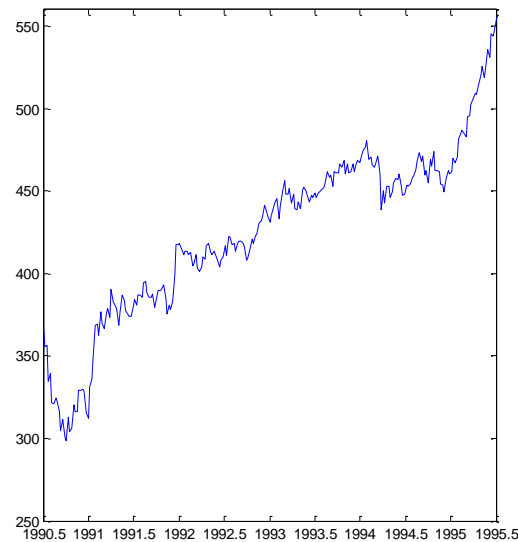


FRACTALES ALEATORIOS

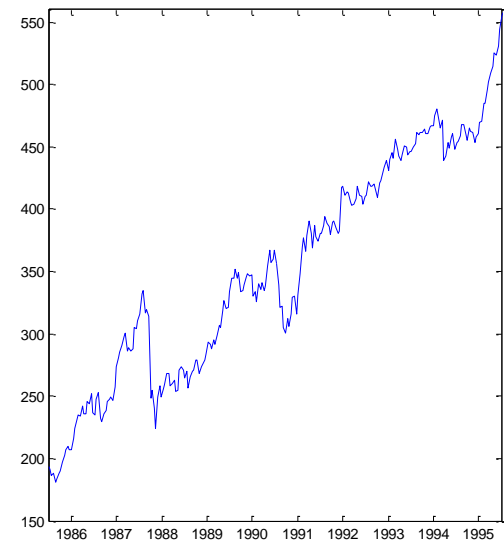
Indice financiero Standards & Poors 500



Un año



Cinco años



diez years

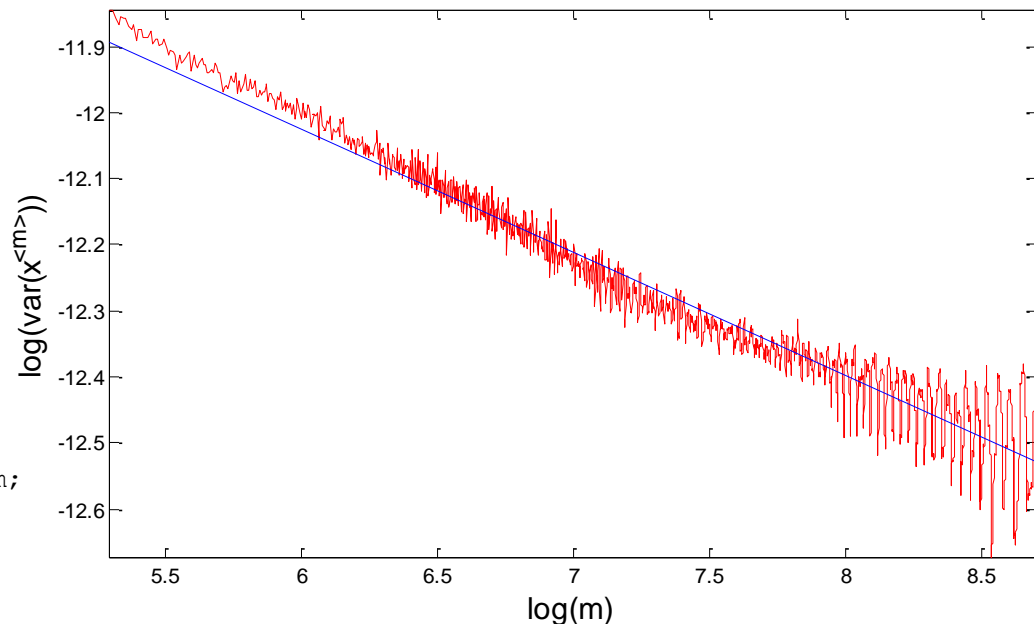
$$Y(t) = a^d Y(at)$$

ESTIMACIÓN DEL PARÁMETRO H

$$X^{<m>}(i) = \frac{1}{m} \sum_{t=1+m \cdot (i-1)}^{m \cdot i} X(t)$$

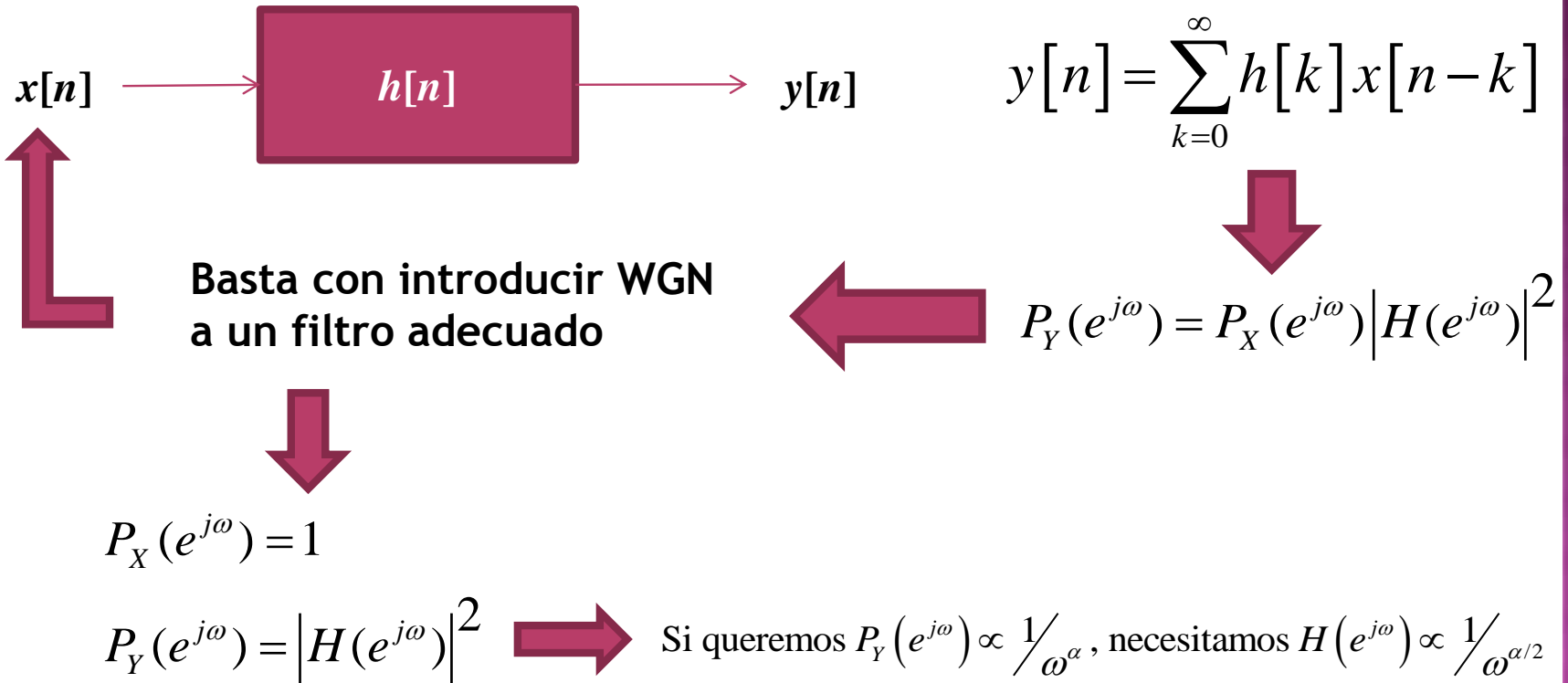
$$\text{Var}[X^{<m>}] = m^{2(H-1)} \text{Var}[X]$$

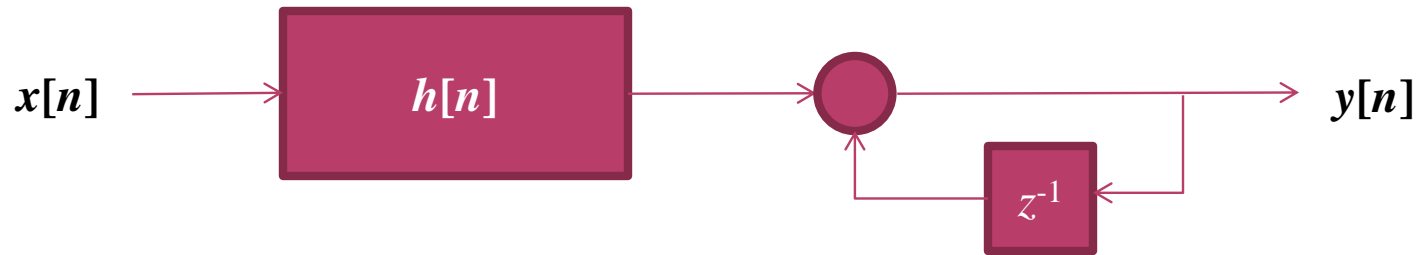
$$\log(\text{Var}[X^{<m>}]) = \log(\text{Var}[X]) - (2(1-H)) \log(m)$$



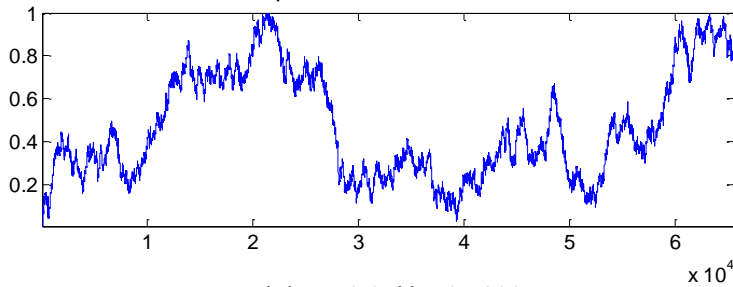
```
load DescargasNeuronales.mat
NP = length(x)-1;
for m=200:6000
    Ns = floor(NP/m);
    y = zeros(Ns,1);
    for i=1:Ns
        y(i) = sum(x(1+m*(i-1):m*i))/m;
    end
    v(m-199) = var(y);
end
m = 200:6000;
p=polyfit(log(m),log(v),1);
plot(log(m),log(v),'r-',[log(m(1)) log(m(end))],[p(2)+log(m(1))*p(1) p(2)+log(m(end))*p(1)],'b-')
H = p(1)/2 + 1;
```

RUIDO 1/F

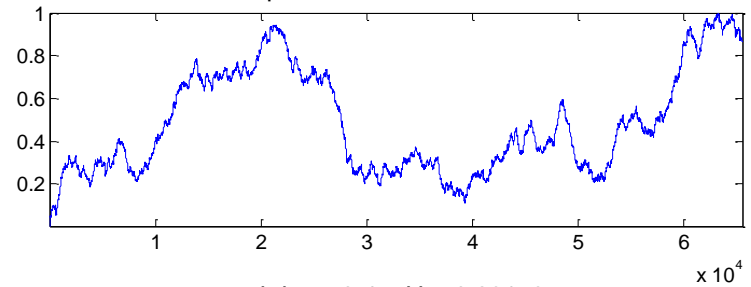




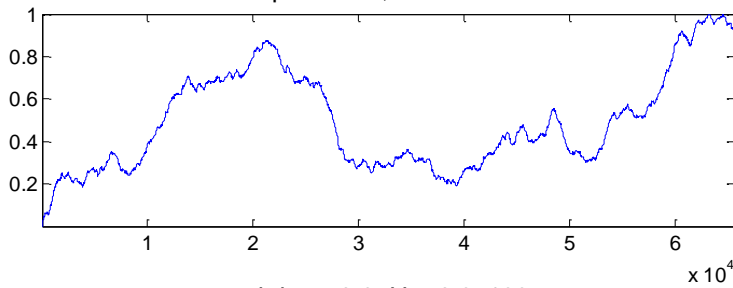
alpha = 0, H = 0.52674



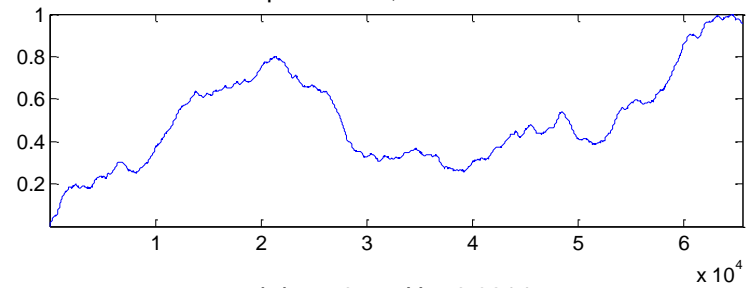
alpha = 0.15, H = 0.66747



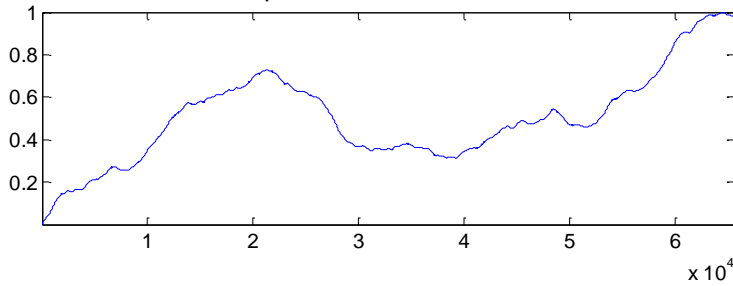
alpha = 0.3, H = 0.7936



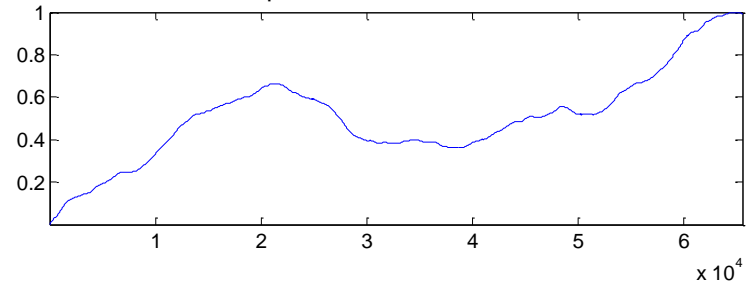
alpha = 0.45, H = 0.89153



alpha = 0.6, H = 0.95199

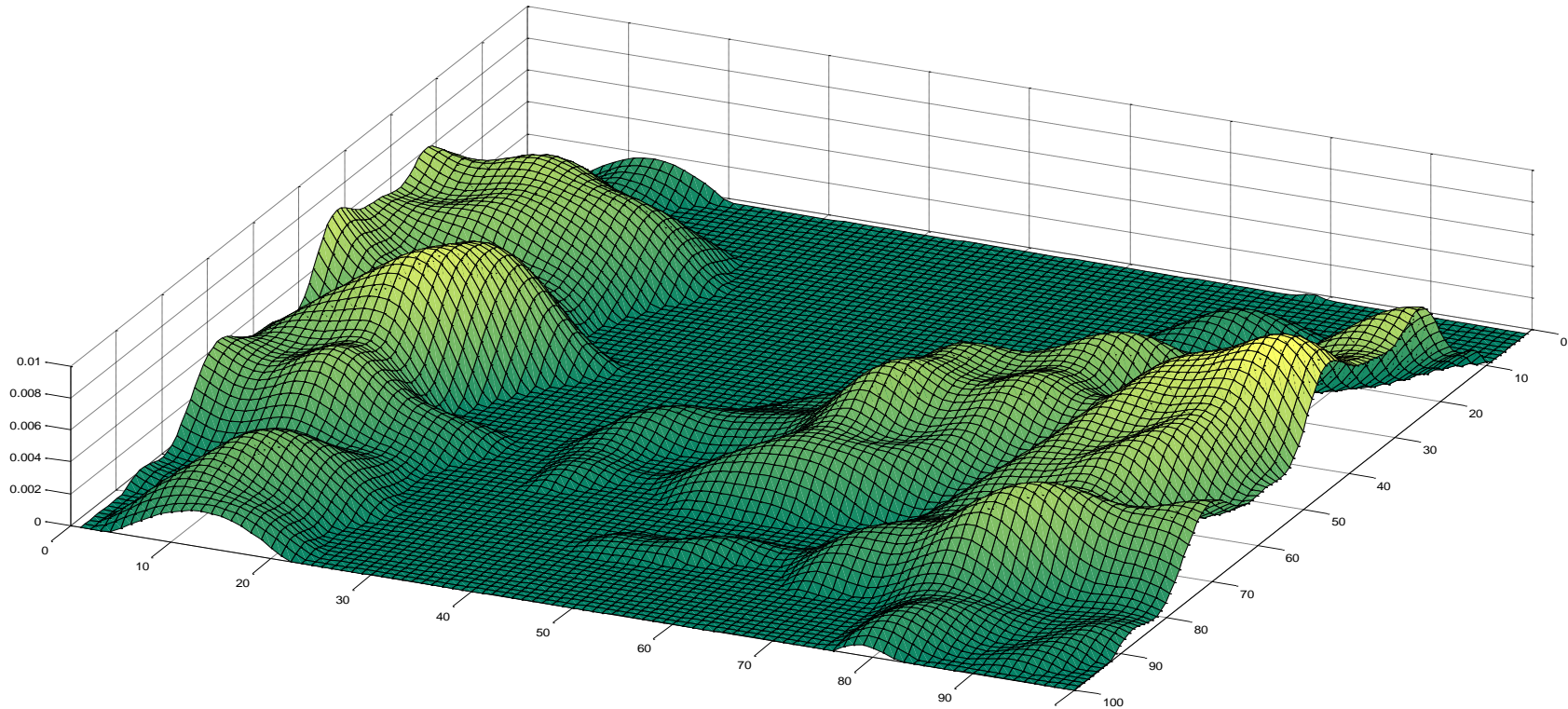


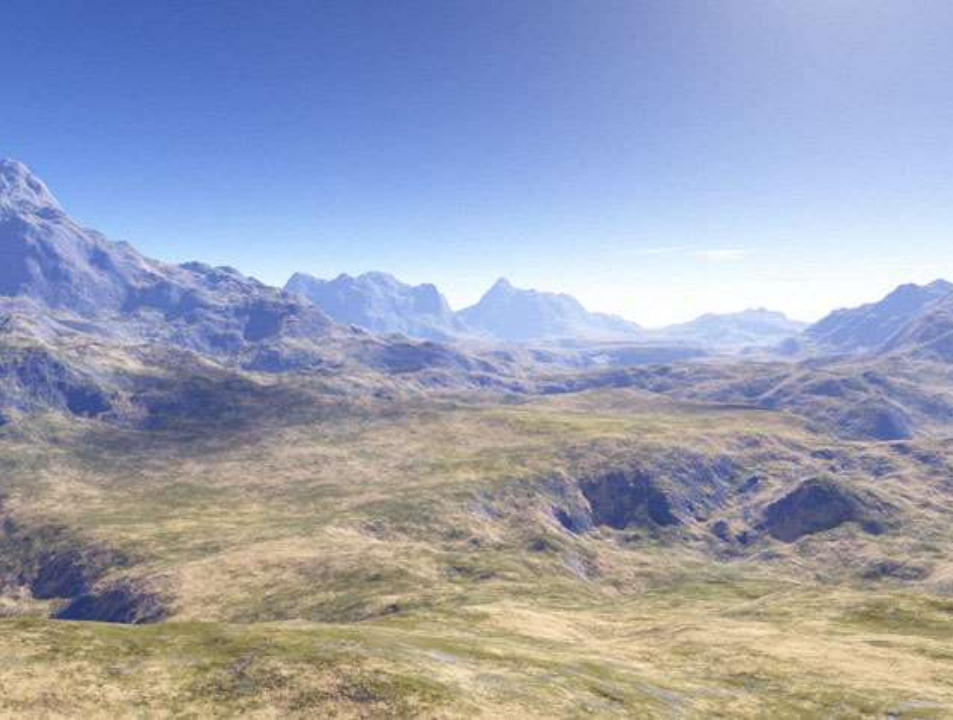
alpha = 0.75, H = 0.98145



PAISAJES FRACTALES

```
N = 80; Nc = 10;           % Tamaño de la grilla, orden del filtro
W = fft2(randn(N));        % ruido blanco gaussiano
V = zeros(N);              % Coeficientes del filtro
for i=1:Nc                  % Ley de potencia
    V(i,1:Nc) = 1./(i^2.1 + (1:Nc).^2.1);
end
y = real(ifft2(V.*W)); % filtrado
my = min(min(y)); MY = max(max(y));
zero = my + 2*(max(max(y)) - min(min(y)))/3; % Nivel del suelo
y = y-zero; y = y.*(y>0); % Recorta al nivel del suelo
surf(y)                    % Grafica
```



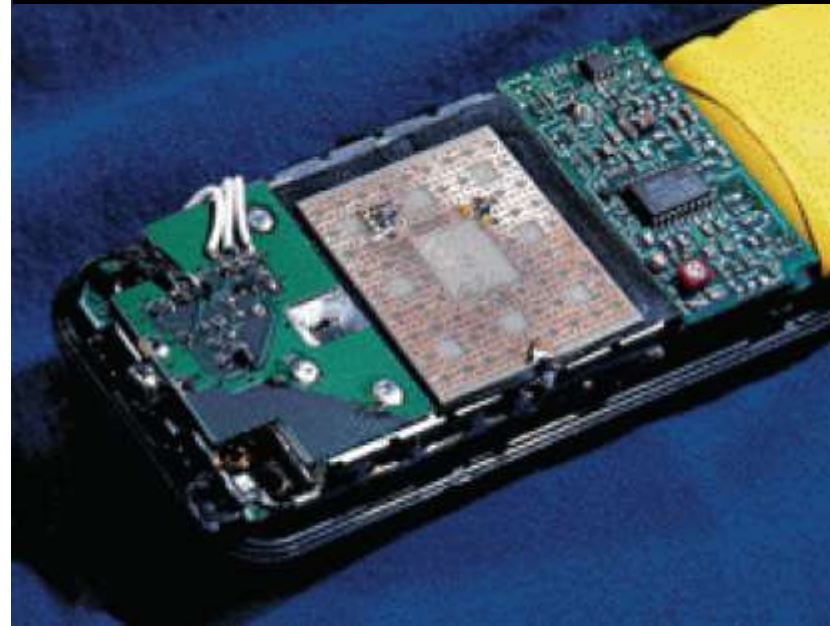
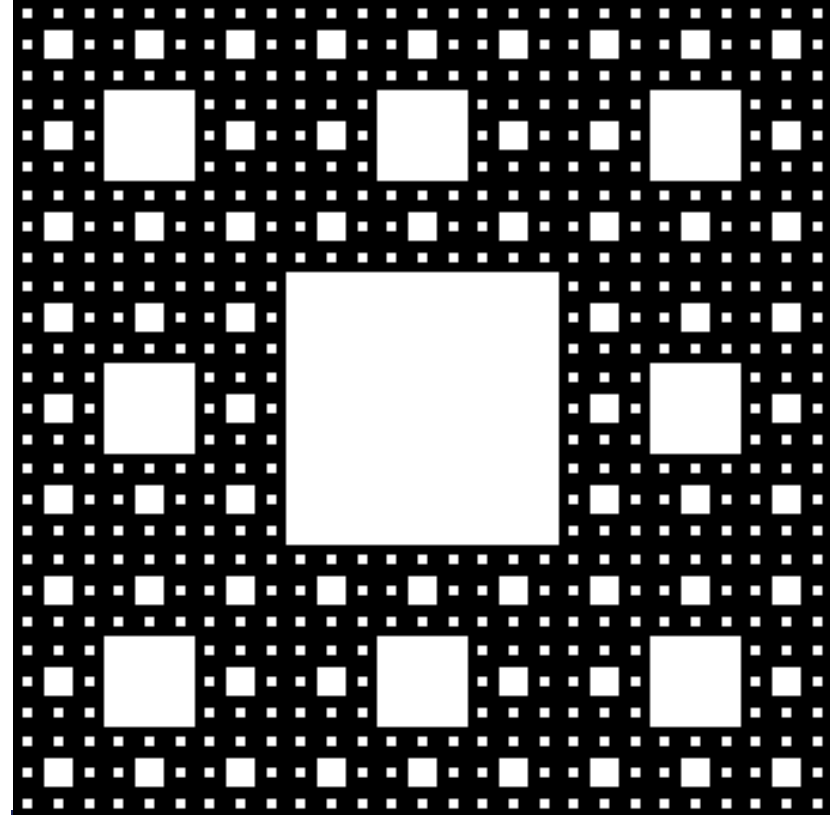


Hasta ahora: Fractales aleatorios o
construidos mediante algoritmos que
reproducen la autosimilitud exacta a
distintas escalas

```
s = 0;  
for k = 1:4  
    s = [s,      s,      s ;  
        s, ones(3^(k-1)), s ;  
        s,      s,      s];  
end  
imagesc(s);
```

Típicamente una antena se construye de acuerdo con la longitud de onda en que debe operar (alambritos de $\lambda/4$). Una antena fractal maximiza el tamaño efectivo de la antena, haciéndola invariante (con las mismas propiedades de radiación) en un amplio rango de escalas de frecuencia.

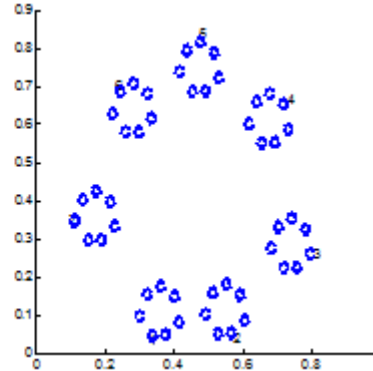
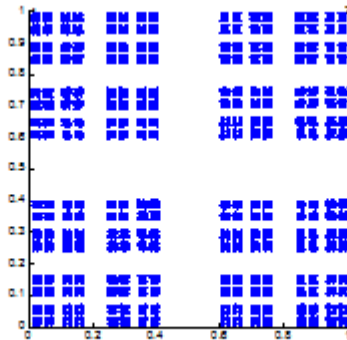
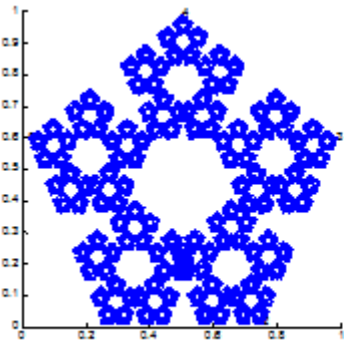
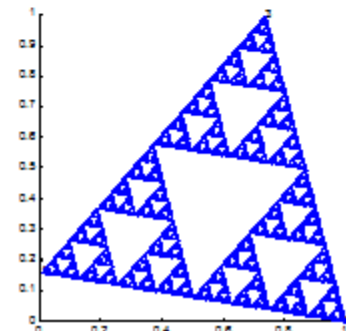
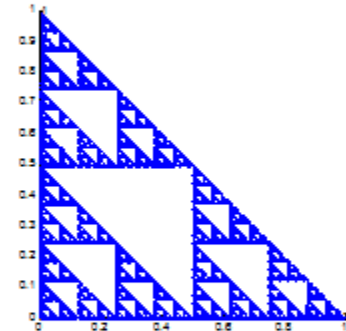
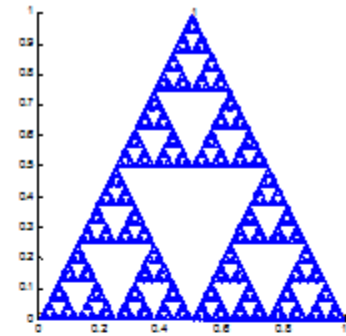
- Compactas
- Múltiples bandas o una banda ultra ancha (UWB)
- Celulares, RFID, etc.



EL JUEGO DEL CAOS

```
NP = 3; NI = 100000; alpha = 1/2;
```

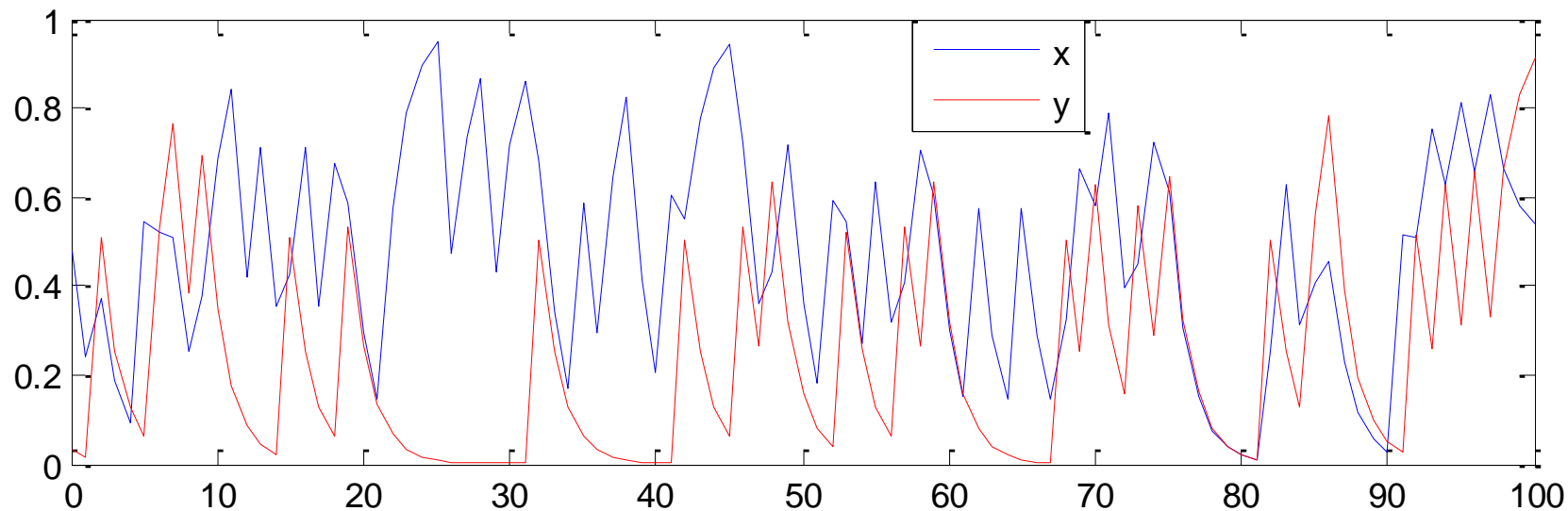
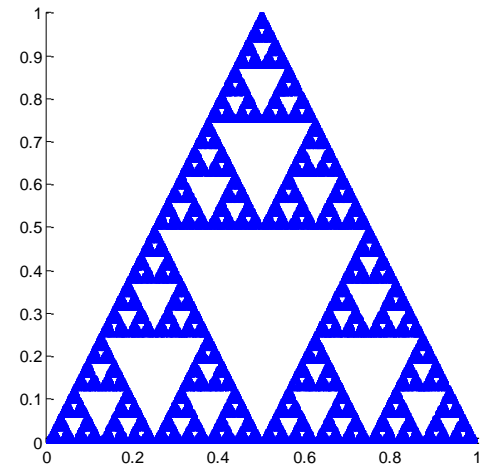
```
P=zeros(2,NP); % Espacio para los vértices de referencia
for i=1:NP % Entra interactivamente cada punto
    t=text(.01,.97,['Seleccione el punto ' int2str(i) '.']);
    p=ginput(1); P(:,i)=p(:); % Almacena el i-ésimo punto
    delete(t); text(p(1),p(2),int2str(i)); % y lo señala en el tablero
end
t=text(.01,.97,'Calculando...'); drawnow
x=[0.5; 0.5]; % Escoge un punto inicial
X=zeros(2,NI); % Espacio para los puntos visitados
for i=-200:NI % Los 200 primeros pasos son para transientes
    r=ceil(NP*rand); % Escoge un vértice destino
    x=x+alpha*(P(:,r)-x); % y se mueve a un punto intermedio
    if i>0, X(:,i)=x; end % Almacena los pasos después del transiente
end % Grafica los puntos visitados
delete(t); line(X(1,:),X(2,:), 'linestyle','none','marker','.', 'markersize',1); axis square
% Otro ejemplo:
% NP = 5; NI = 100000; alpha = 3/5
```



Sistemas Dinámicos!

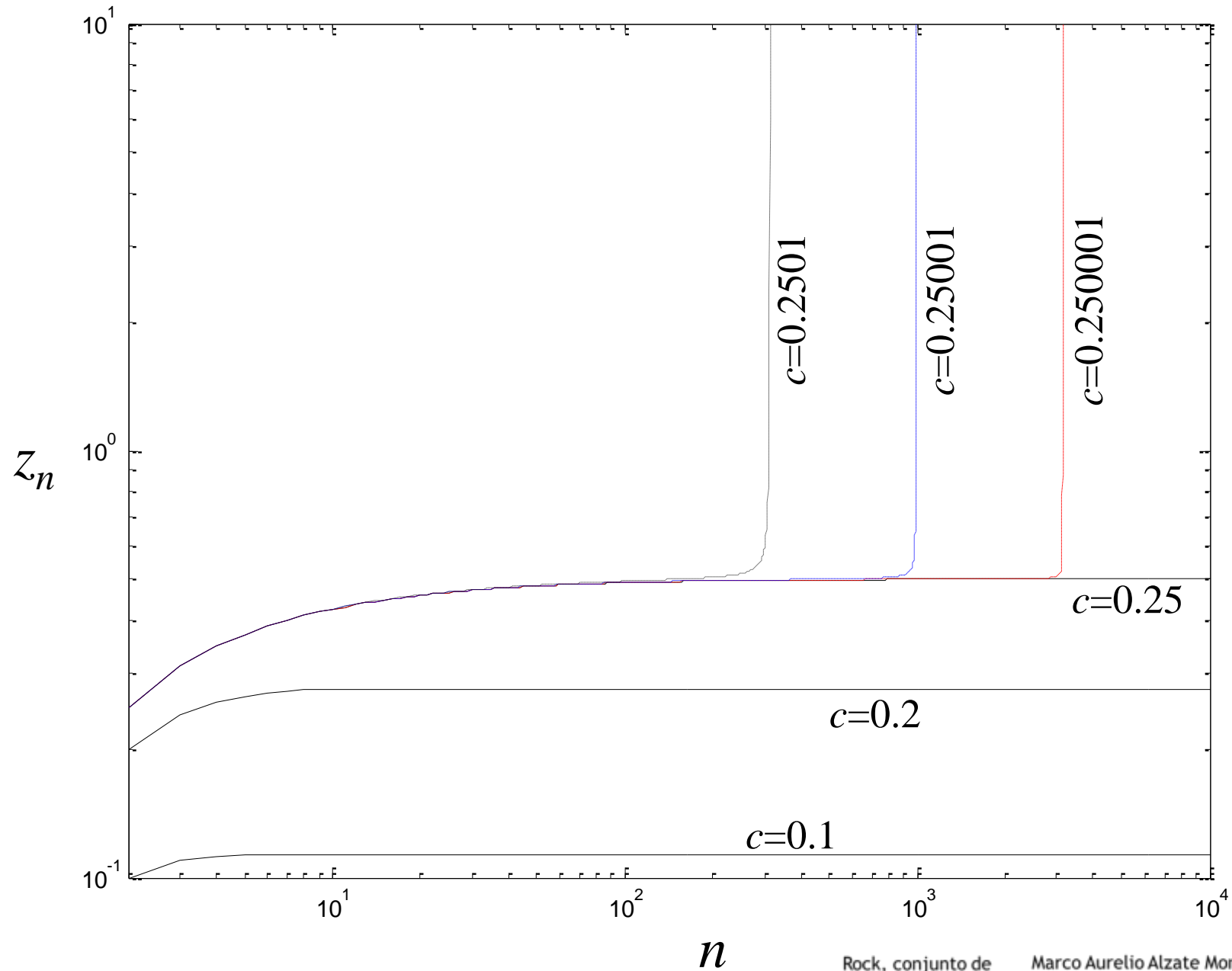
EL JUEGO DEL CAOS

$$x_n = \frac{1}{2}(Px_i - x_{n-1}), \quad y_n = \frac{1}{2}(Py_i - y_{n-1})$$

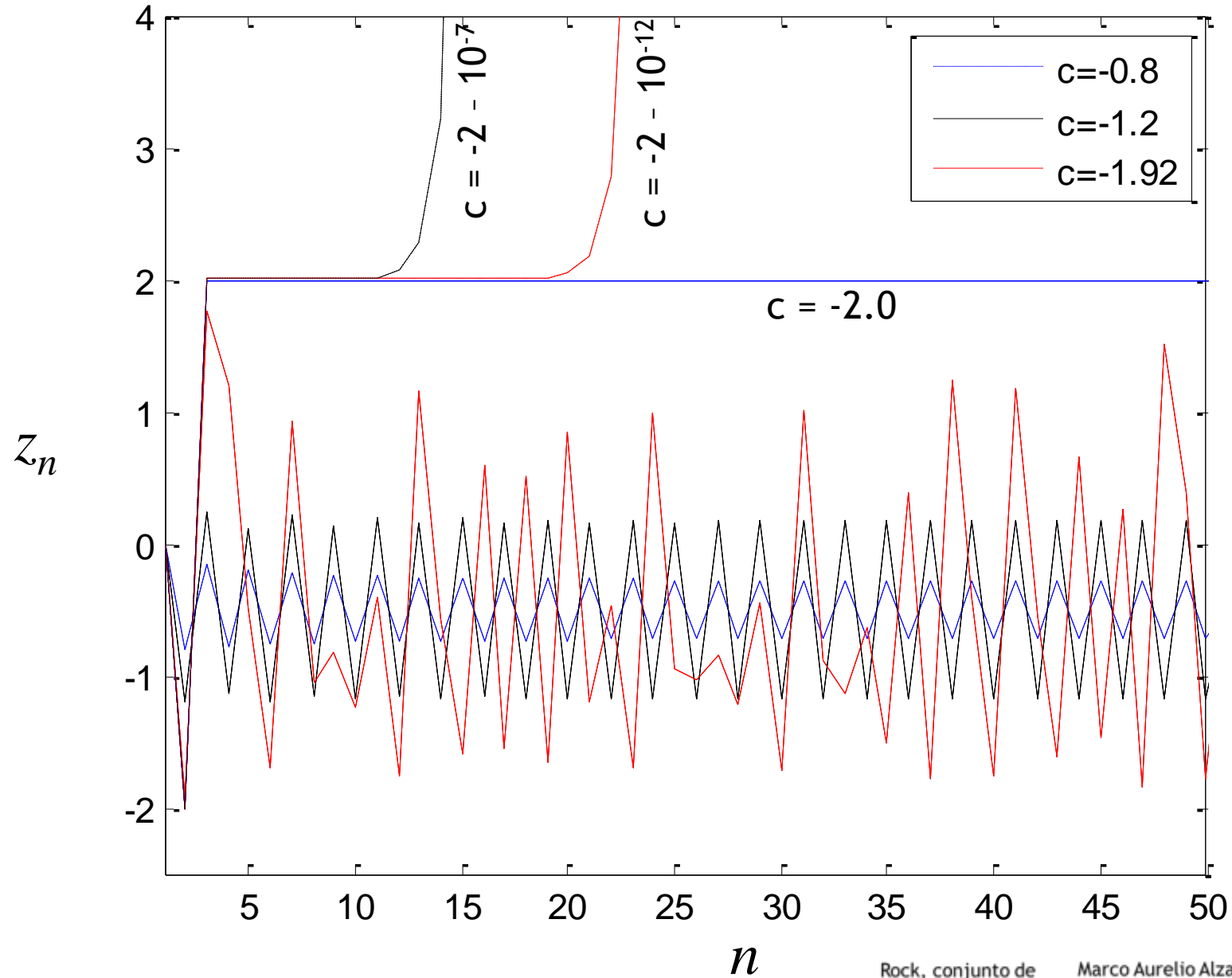


Un sistema dinámico en el que el estado (las coordenadas (x,y)) evoluciona con el tiempo (el índice n)

$$z_n = c + z_{n-1}^2, \quad z_0 = 0$$



$$z_n = c + z_{n-1}^2, \quad z_0 = 0$$



PARTE 6

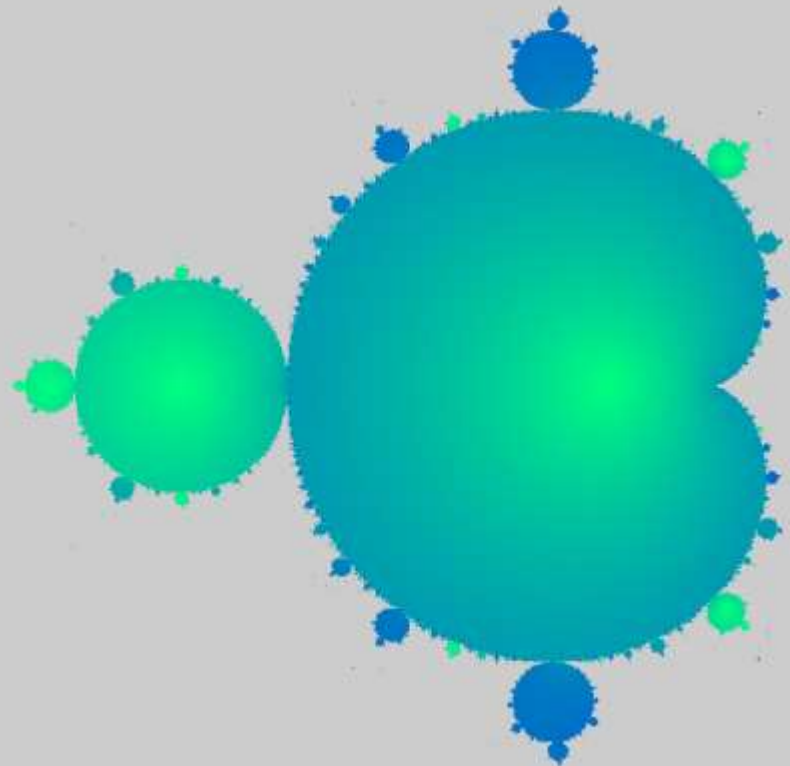
- ◉ And he looked below the storm
and saw a vision in his head
A bulbous pointy form
He picked his pencil up
and he wrote his secret down:
- ◉ Just take a point called Z in the complex plane
Let Z_1 be Z squared plus C
And Z_2 is Z_1 squared plus C
And Z_3 is Z_2 squared plus C and so on
If the series of Z 's should always stay close to Z
And never trend away
That point is in the Mandelbrot Set

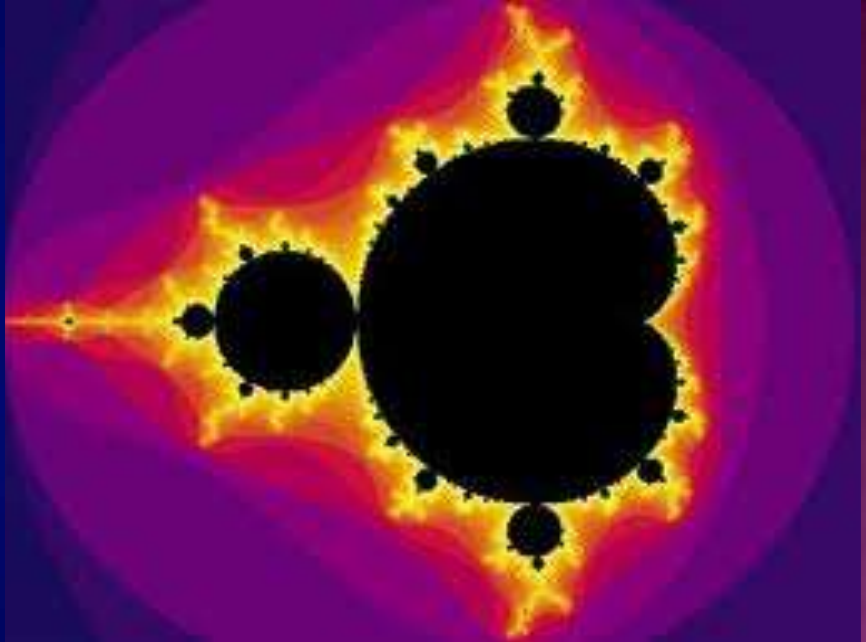
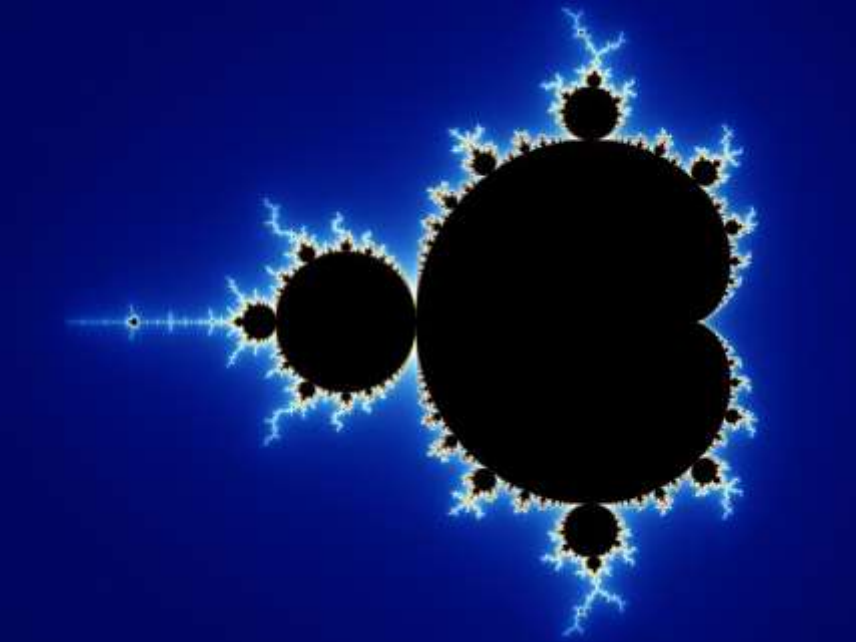
La región donde $\{z_n = c + z_{n-1}^2, \quad z_0 = 0\}$ no escapa a ∞ en \mathbb{R} es $[-2.0 \ 0.25]$

¿Cuál es la región donde $\{z_n = c + z_{n-1}^2, \quad z_0 = 0\}$ no escapa a ∞ en \mathbb{C} ?

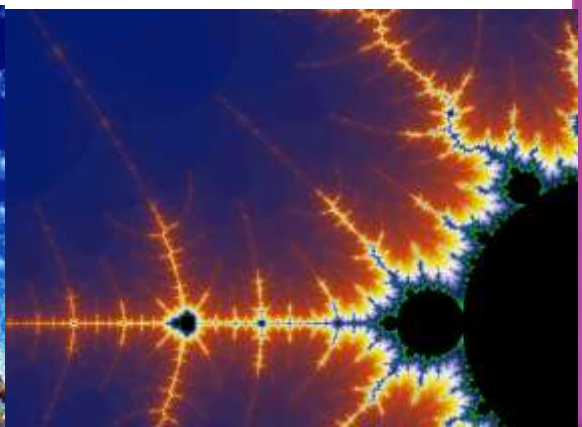
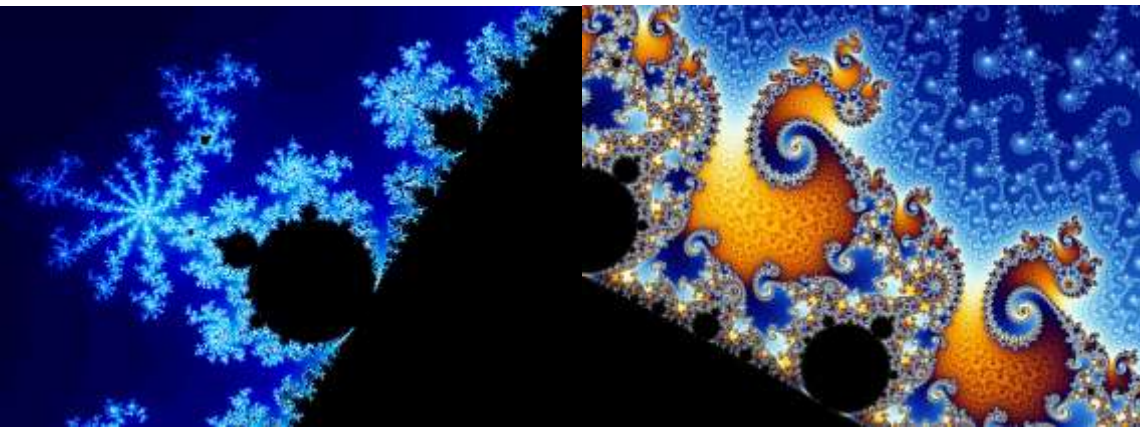
Repitamos el proceso anterior : escojamos 1'440.000 valores de c en un rectángulo de 2×2 y coloreemos c con color 1 si $|Z_{256}|=0$, con color 0 si $|Z_{256}|=\infty$ y otros colores en $(0,1)$ para valores intermedios de $|Z_{256}|$.

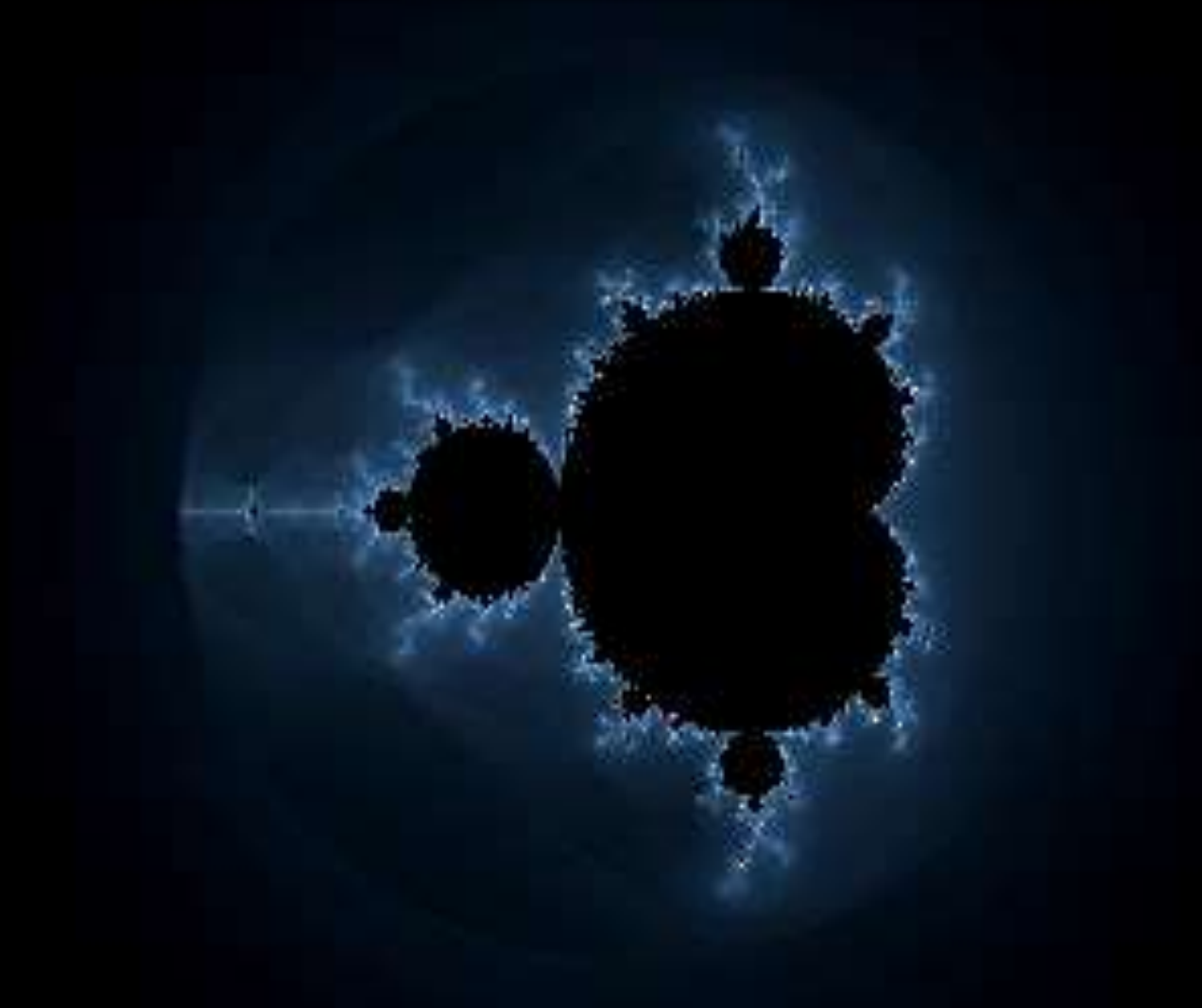
```
x      = linspace(-1.5,0.5,1200)
y      = linspace(-1,1,1200);
[X,Y] = meshgrid(x,y);
Z      = zeros(1200);
C      = X+i*Y;
for k = 1:256
    Z = Z.^2+C;
end
colormap winter(256);
pcolor(x,y,exp(-abs(Z)));
shading flat;
axis('square','equal','off');
```





Mandelbrot Set you're a Rorschach Test on fire
You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire





You're one badass fucking fractal



Pathological monsters!
Cried the terrified mathematician
Every one of them is a splinter in my eye
I hate the Peano Space and the Koch Curve
I fear the Cantor Ternary Set
And the Sierpinski Gasket makes me want to cry

And a million miles away a butterfly flapped its wings
On a cold November day
A man named Benoit Mandelbrot was born

His disdain for pure mathematics
And his unique geometrical insights
Left him well equipped to face those demons down
He saw that infinite complexity
Could be described by simple rules
He used his giant brain to turn the game around
And he looked below the storm
And saw a vision in his head, A bulbous pointy form
He picked his pencil up and he wrote his secret down

Just take a point called Z in the complex plane
Let Z_1 be Z squared plus C
And Z_2 is Z_1 squared plus C
And Z_3 is Z_2 squared plus C and so on
If the series of Z 's should always stay close to Z
And never trend away
That point is in the Mandelbrot Set

Mandelbrot Set you're a Rorschach Test on fire
You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire
You're one badass fucking fractal
And you're just in time to save the day

Sweeping all our fears away
You can change the world in a tiny way

Mandelbrot's in heaven, at least he will be when he's dead
Right now he's still alive and teaching math at Yale
He gave us order out of chaos
He gave us hope where there was none
And his geometry succeeds where others fail
If you ever lose your way, a butterfly will flap its wings
From a million miles away, a little miracle will come
To take you home

Just take a point called Z in the complex plane
Let Z_1 be Z squared plus C
And Z_2 is Z_1 squared plus C
And Z_3 is Z_2 squared plus C and so on
If the series of Z 's should always stay close to Z
And never trend away
That point is in the Mandelbrot Set

Mandelbrot Set you're a Rorschach Test on fire
You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire
You're one badass fucking fractal
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
Go on change the world in a tiny way
Come on change the world in a tiny way

Mandelbrot Set
by Jonathan Coulton

El error es evidente:

Just take a point called Z in the complex plane

Let Z_1 be Z squared plus C

And Z_2 is Z_1 squared plus C

And Z_3 is Z_2 squared plus C and so on

If the series of Z 's should always stay close to Z

And never trend away

That point is in the Mandelbrot Set

```
x      = linspace(-1.5,0.5,1200);
y      = linspace(-1,1,1200);
[X,Y]  = meshgrid(x,y);
Z      = zeros(1200);
C      = X+i*Y;
for k = 1:256
    Z = Z.^2+C;
end
colormap winter(256);
pcolor(x,y,exp(-abs(Z)));
shading flat;
axis('square','equal','off');
```

Lo que debió decir:

Just take a point called C in the complex plane

Let Z_1 be $Zero$ squared plus C

And Z_2 is Z_1 squared plus C

And Z_3 is Z_2 squared plus C and so on

If the series of Z 's should always stay close to $Zero$

And never trend away

The point C is in the Mandelbrot Set

Lo que Jonathan describió fue un “conjunto relleno” de Julia:

Given a complex number C

Just take a point called Z_0 in the complex plane

Let Z_1 be Z_0 squared plus C

And Z_2 is Z_1 squared plus C

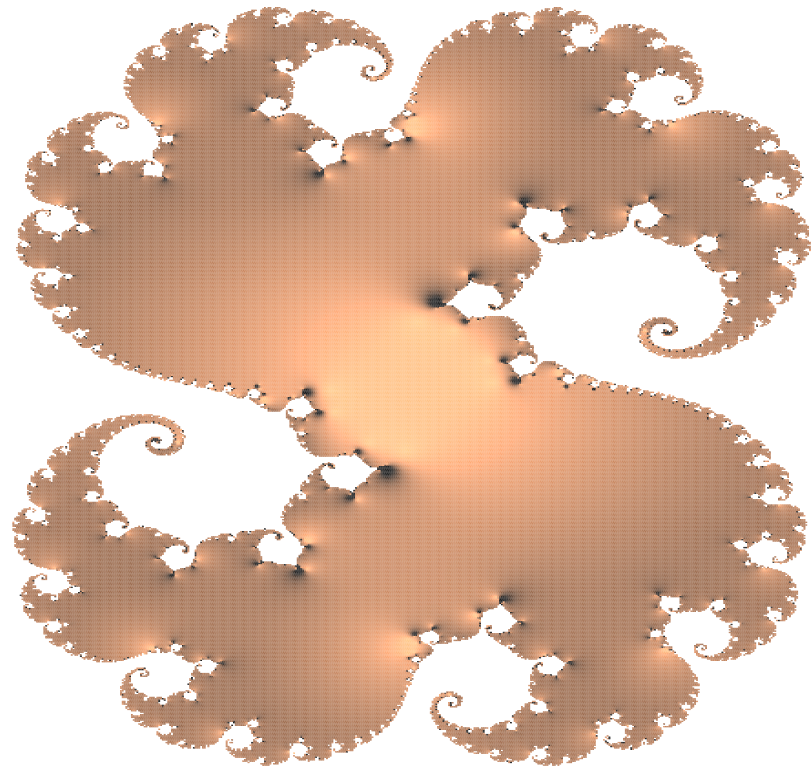
And Z_3 is Z_2 squared plus C and so on

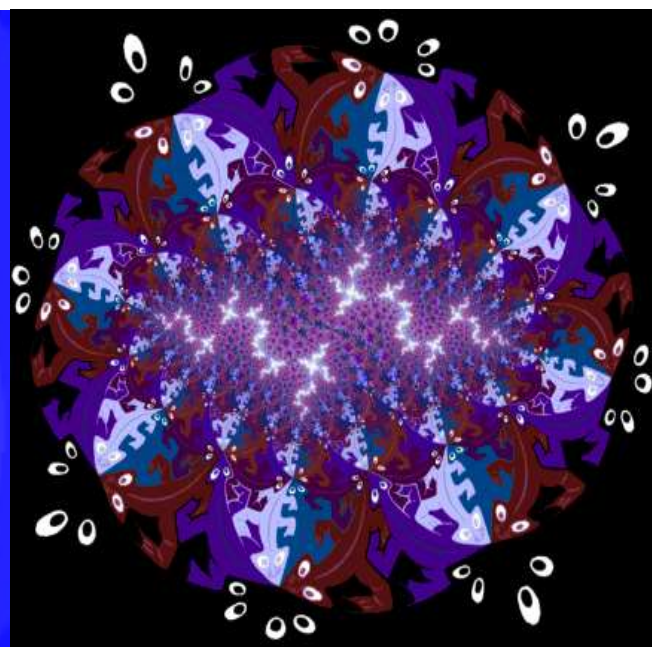
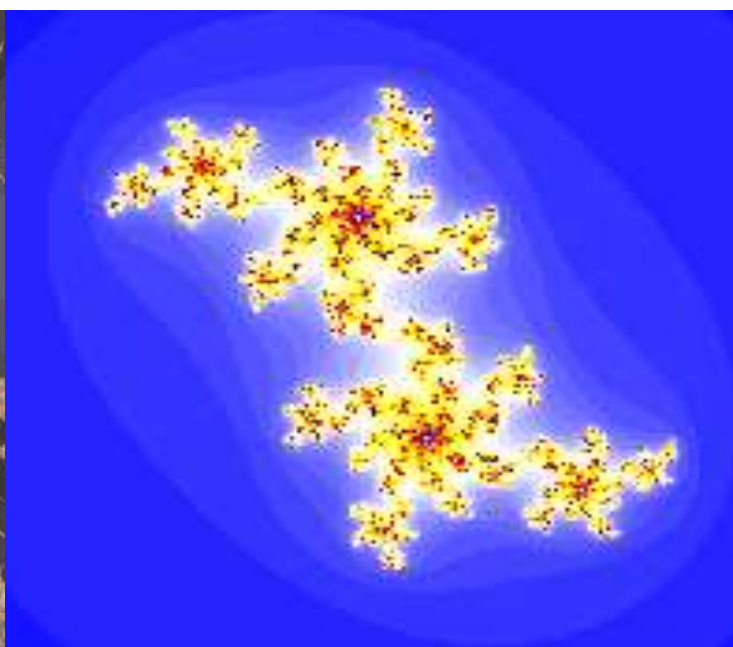
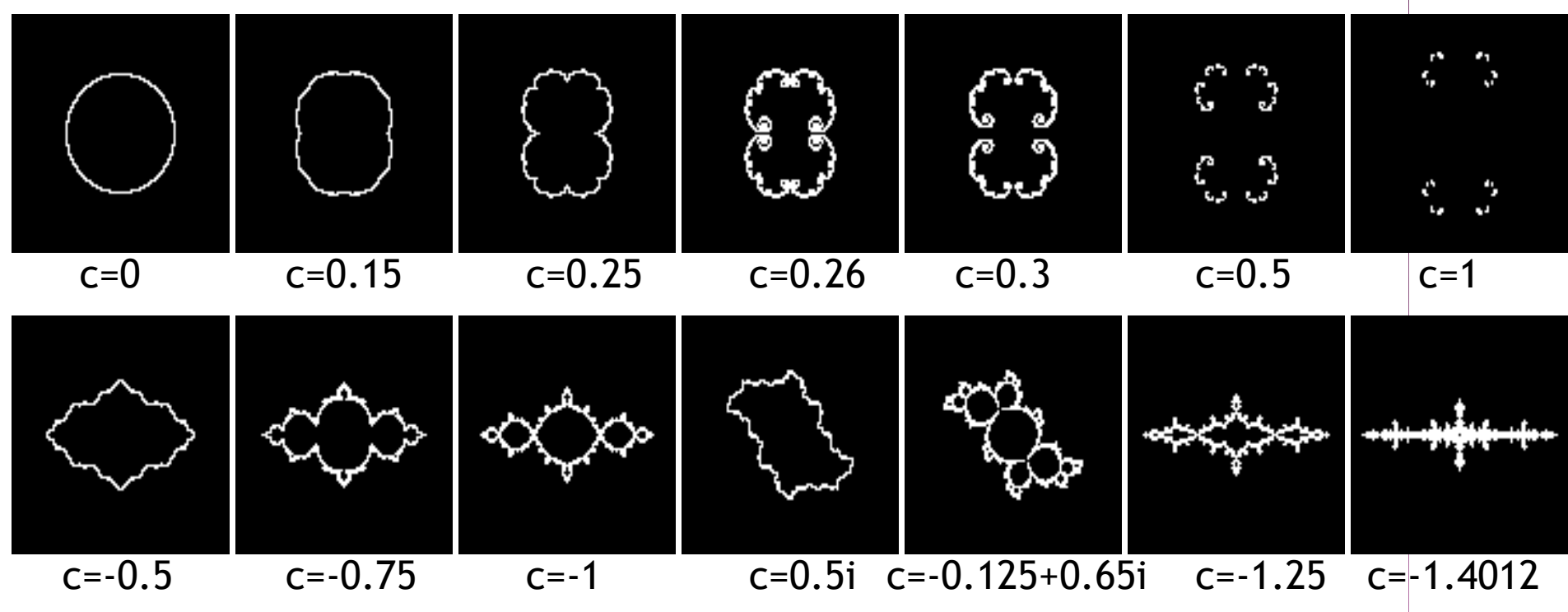
If the series of Z 's should always stay close to Z_0

And never trend away

Then the point Z_0 is in the filled-in Julia Set for C

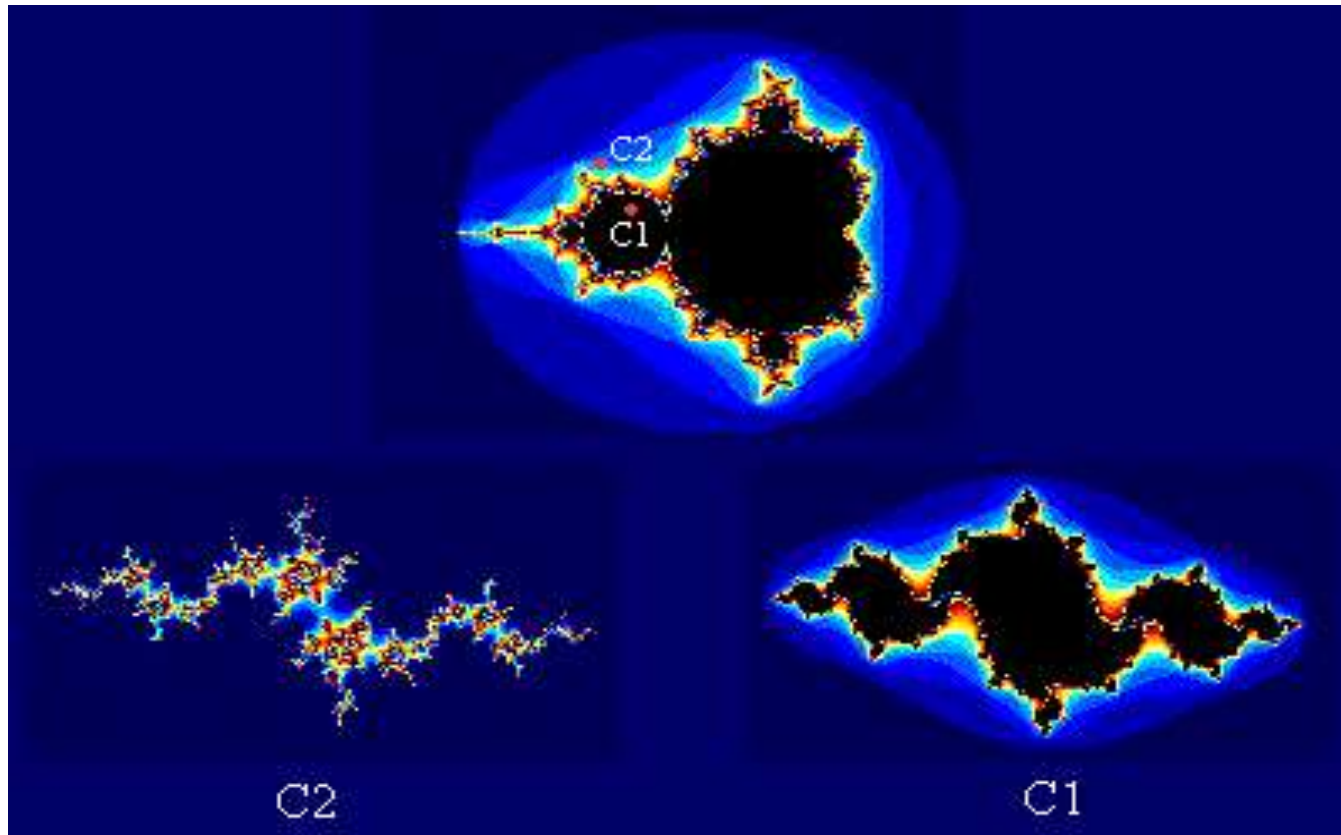
```
c = 0.27334-0.00742*i;  
x=linspace(-1.5,1.5,800);  
y=linspace(-1.5,1.5,800);  
[X,Y]=meshgrid(x,y);  
Z=X+i*Y;  
for k=1:60  
    Z=Z.^2+c;  
end  
colormap copper(256)  
pcolor(x,y,exp(-abs(Z)));  
shading flat;  
axis('square','equal','off');
```





De hecho, la definición formal del conjunto de Mandelbrot es:

Conjunto de parámetros c para los cuales el correspondiente conjunto de Julia es conectado



Formalmente:

Sea $f : \mathbb{C} \rightarrow \mathbb{C}$ un polinomio de grado $n \geq 2$

Sea $f^k(w)$ la k -ésima composición iterada de f , $f^k(w) = f(f(\dots(f(w))\dots))$

Los conjuntos de Julia se definen en término del comportamiento de las iteraciones $f^k(z)$ para valores grandes de k

Primero definimos el "conjunto relleno" de Julia del polinomio f

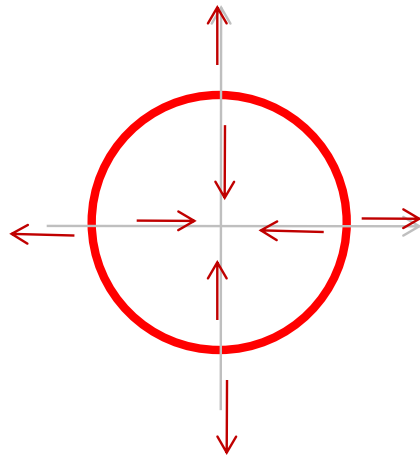
$$K(f) = \left\{ z \in \mathbb{C} : f^k(z) \not\rightarrow \infty \right\}$$

Ahora podemos definir el conjunto de Julia para f como la frontera de $K(f)$:

$$J(f) = \partial K(f) = \left\{ z \in \mathbb{C} : \forall \varepsilon \in \mathbb{R} \exists w, v \in \mathbb{C} \text{ s.t. } |z - w| < \varepsilon, |z - v| < \varepsilon, f^k(w) \not\rightarrow_k \infty \text{ and } f^k(v) \rightarrow_k \infty \right\}$$

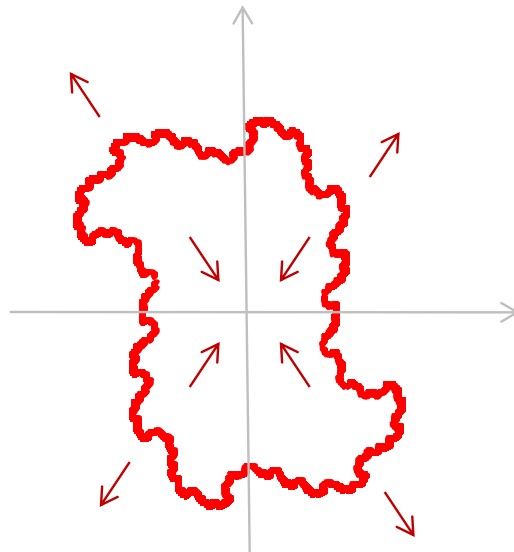
Típicamente: $f(z) = c + z^2$

Por ejemplo, $c=0$: $f(z) = z^2 \Rightarrow f^k(z) = z^{2^k} \rightarrow \begin{cases} 0 & |z_0| < 1 \\ \text{en el círculo } |z|=1 & |z_0| = 1 \\ \infty & |z_0| > 1 \end{cases}$



!No es un fractal! ☺

Si c se incrementa un poco, $c=0.3 + 0.3j$, sigue siendo cierto que $f^k(z) \rightarrow w \approx 0$ para z pequeña, y $f^k(z) \rightarrow \infty$ para z grande. El conjunto de Julia para este valor de c es la frontera entre estos dos tipos de comportamiento.



!Ahora sí es un fractal! ☺

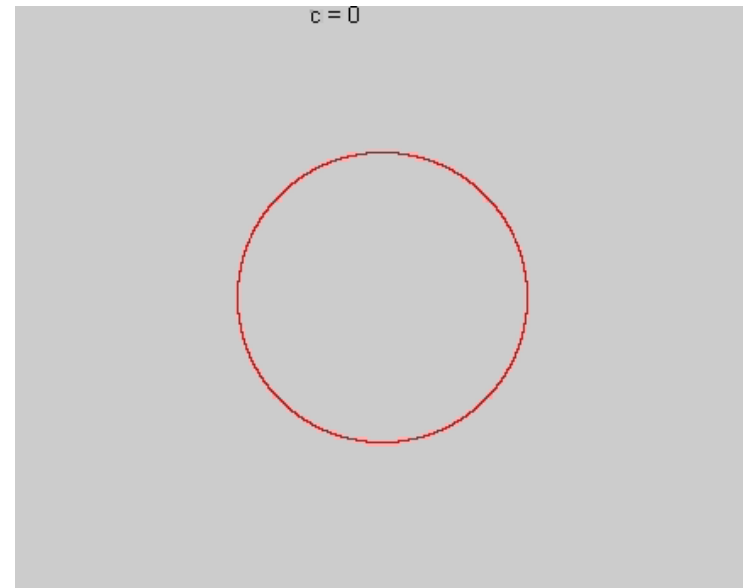
Una forma rápida de dibujar la frontera

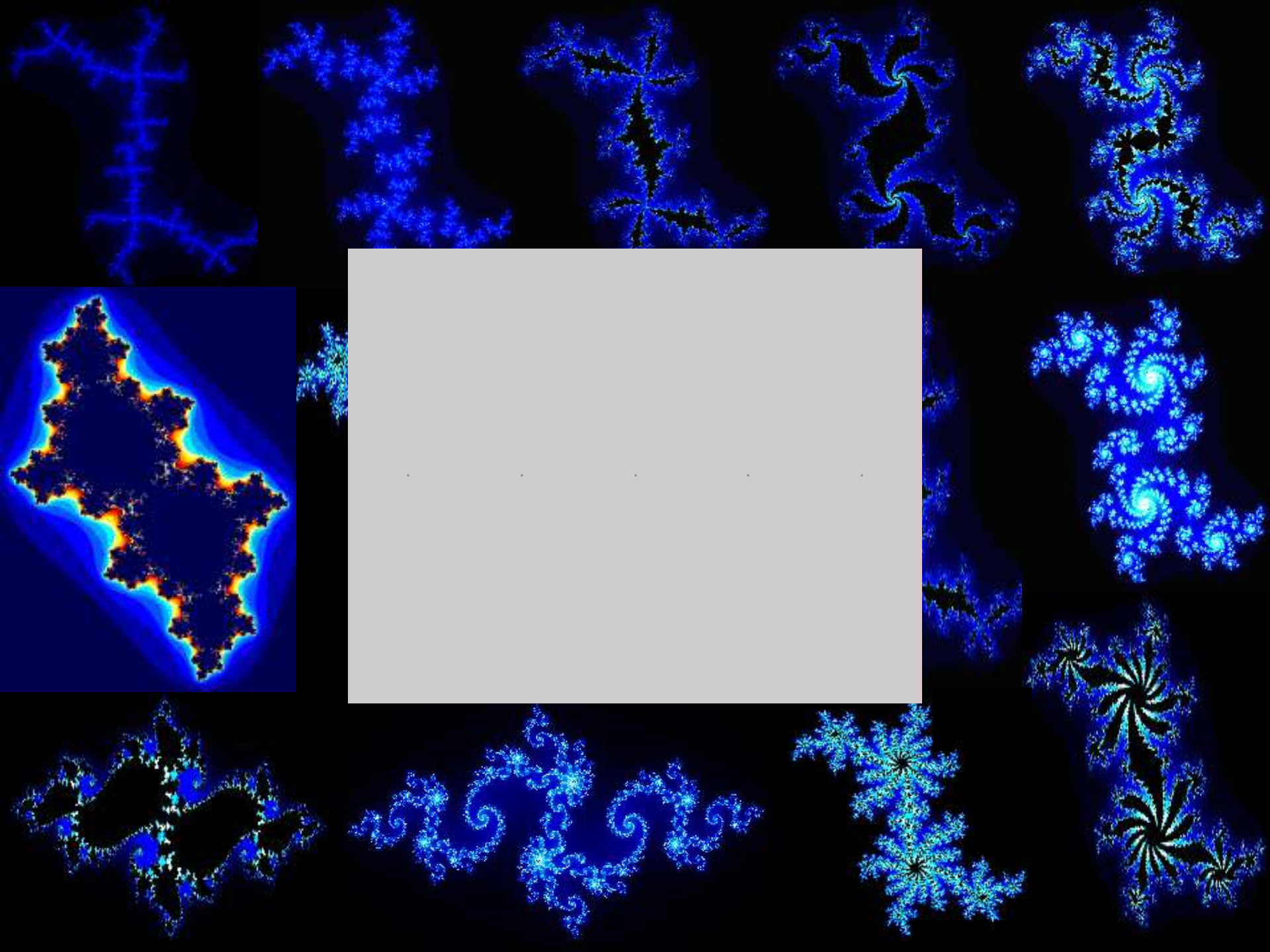
$J(c)$ es invariante, hacia adelante y hacia atrás, con respecto a $f(z) = z^2 + c$

$$J(c) = f(J(c)) = f^{-1}(J(c))$$

$$\text{Si } z \in J(c), \quad \pm \sqrt{z - c} \in J(c)$$

```
c = 0; n = 0;
while(c > -2)
    x = zeros(50000,1);
    r = 2*ceil(2*rand(50000,1))-3;
    for i=2:50000
        x(i) = r(i)*sqrt(x(i-1)-c);
    end
    plot(x(50:end), 'r.', 'MarkerSize', 2)
    n = n+1; F(n) = getframe;
    c = c - 0.05;
end
movie2avi(F, 'JuliaMovieRealC', 'compression', 'Cinepak')
```

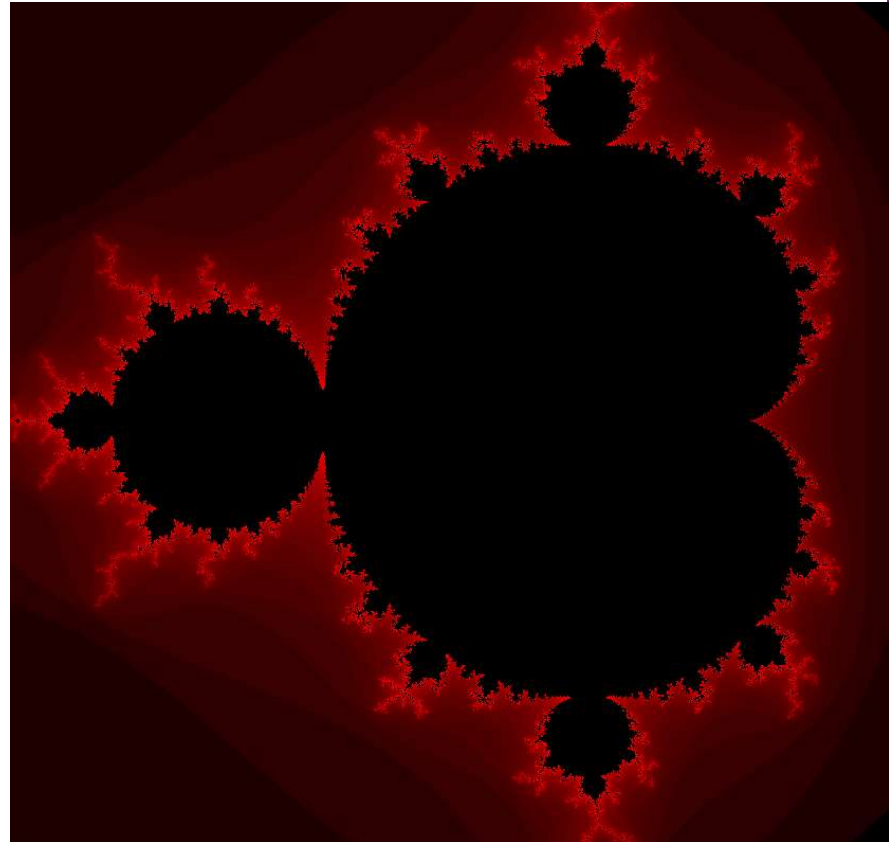




Otra forma de dibujar el conjunto de Mandelbrot:

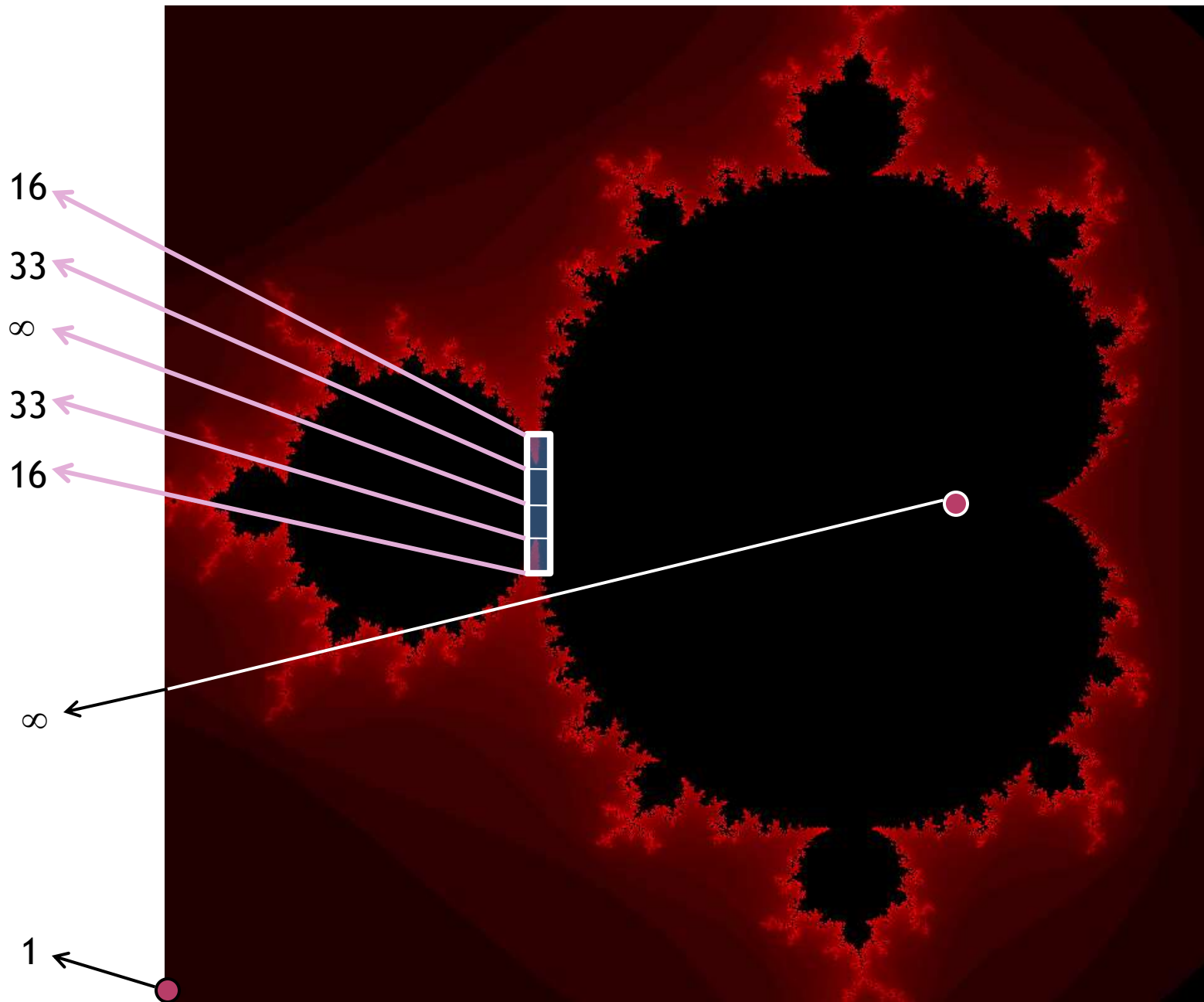
Colorear cada punto según el número de iteraciones para escapar a ∞

```
x      = linspace(-1.5,0.6,1000) ;
y      = linspace(-1,1,1000) ;
[X,Y] = meshgrid(x,y) ;
Z      = zeros(1000) ;
C      = X+i*Y;
for i = 1:1000
    for j=1:1000
        z = 0;
        for t=1:256
            z = z.^2+C(i,j) ;
            if abs(z)>2, break; end
        end
        Z(i,j)=t-1;
    end
end
End
pcolor(x,y,Z) ;
```



[Para el color : shading flat; axis('square','equal','off');cm = zeros(256,3); cm(1:40,1) = (0:39)'/39; colormap(cm); brighten(0.5)]

¿Cuántas iteraciones se necesitan para llegar a $|z| > 2$?



¿Cuántas iteraciones se necesitan para llegar a $|z|>2$?

```
im = 1;  
epsilon = zeros(7,1);  
iteraciones = zeros(7,1);  
for k = 1:7  
    im = im/10;  
    c = -0.75 + im*1j;  
    z = 0;  
    for t=1:100000000  
        z = z.^2+c;  
        if abs(z)>2, break; end  
    end  
    epsilon(k) = im;  
    iteraciones(k) = t;  
end
```

Epsilon	Iteraciones
0.10000000	33
0.01000000	315
0.00100000	3143
0.00010000	31417
0.00001000	314160
0.00000100	3141593
0.00000001	31415927

¿Qué hace π escondido en el conjunto de Mandelbrot?

¿Cuántos secretos más se esconden en él?



HE GAVE US ORDER
OUT OF CHAOS

HE GAVE US HOPE
WHERE THERE WAS
NONE

AND HIS GEOMETRY
SUCCEEDS WHERE
OTHERS FAIL

GO ON CHANGE THE WORLD IN A TINY WAY



MUCHAS GRACIAS

Esta presentación se encuentra en:
<http://comunidad.udistrital.edu.co/malzate>