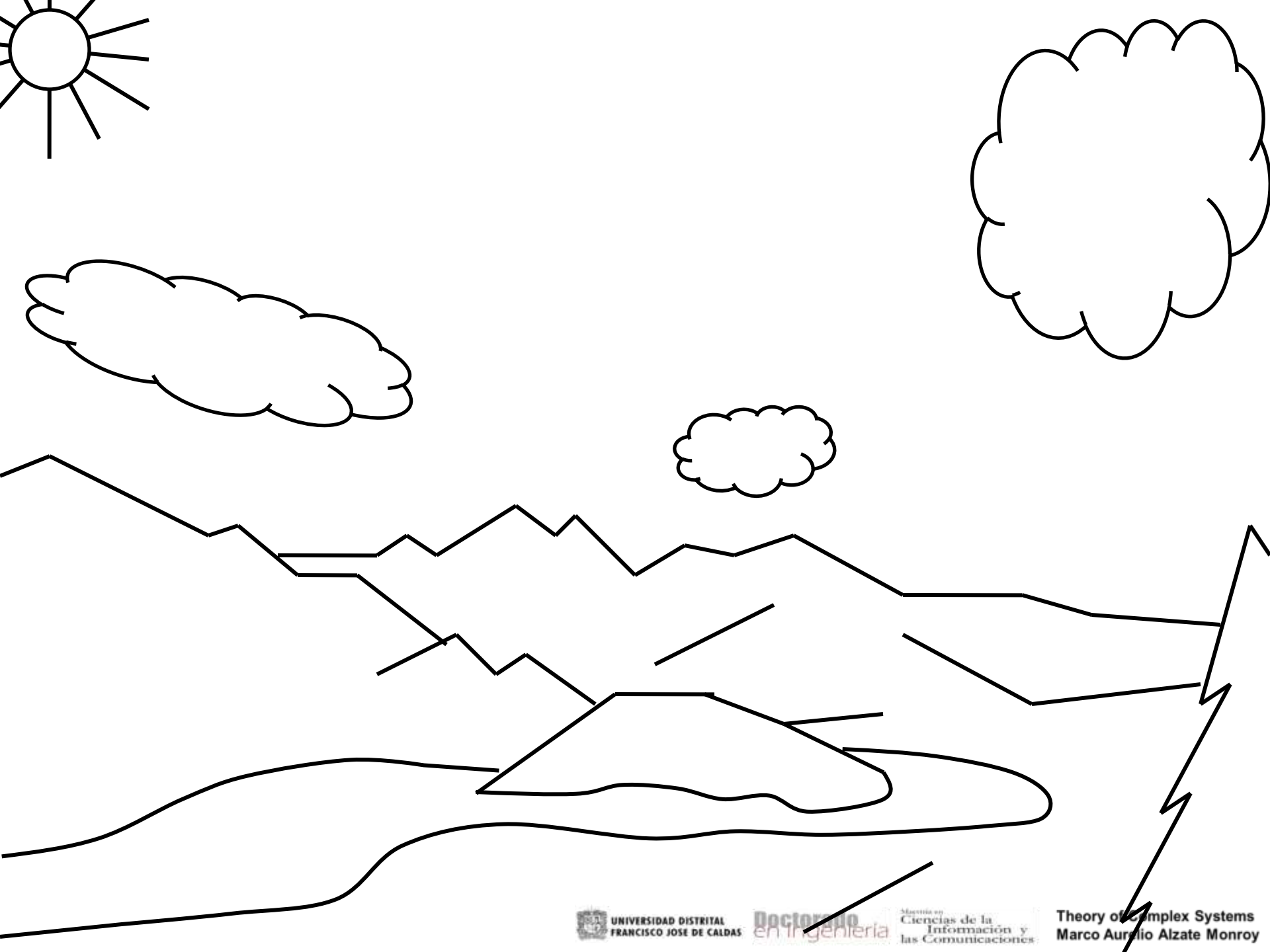
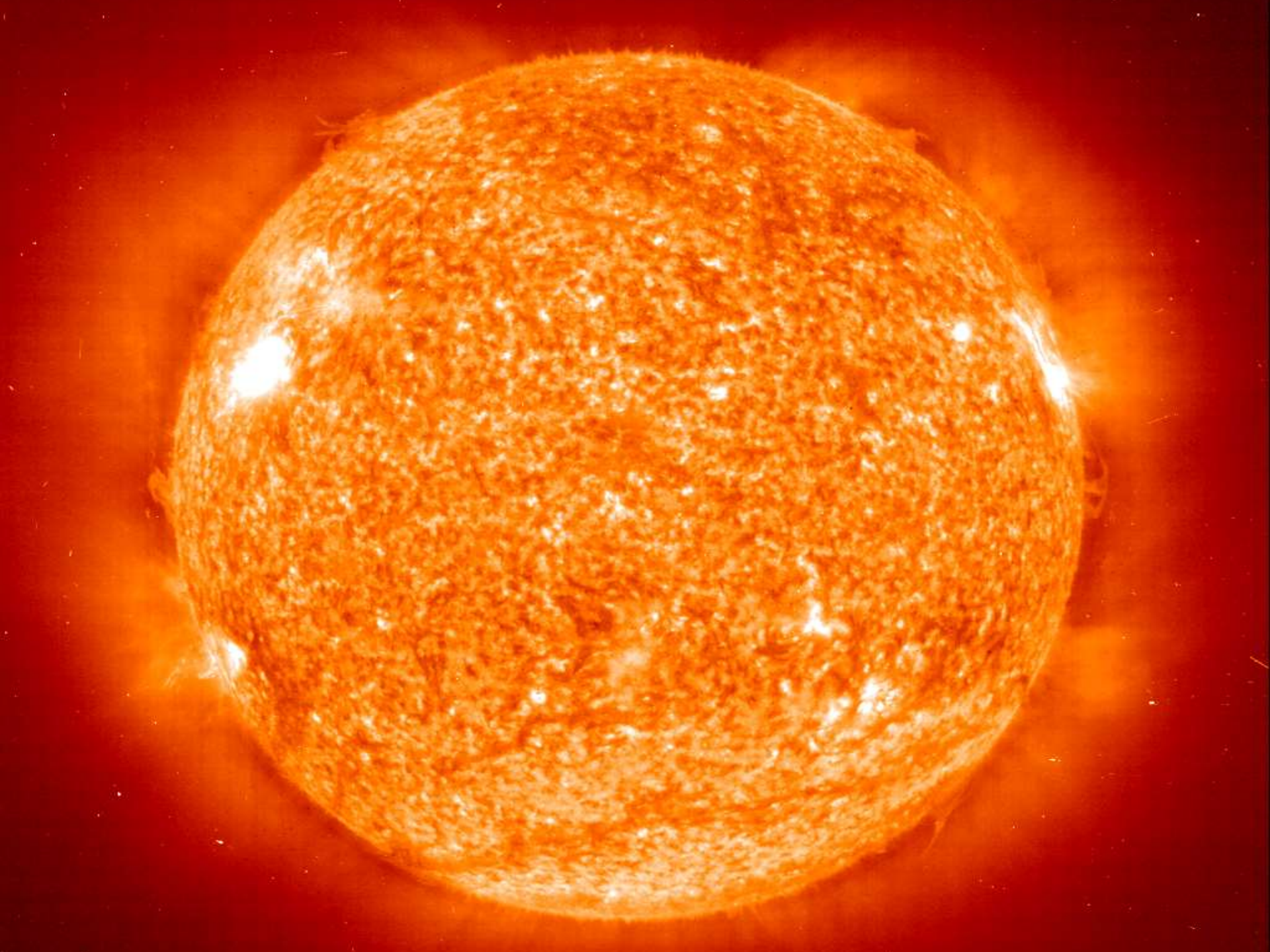
Theory of Complex Systems

Marco Aurelio Alzate Monroy

Deterministic Fractals

Doctoral Program in Engineering
Information and Knowledge Sciences
District University Francisco José de Caldas





In nature, smooth shapes are extremely rare, in fact, quite exceptional. Mountains are not triangles and clouds are not spheres.

Benoit Mandelbrot
III Encuentro Interuniversitario de Complejidad
Bogotá, 2008

Fractals

MISSISSIPPI RIVER DELTA

Turbid waters spill out into the Gulf of Mexico where their suspended sediment is deposited to form the Mississippi River Delta. Beautiful Satellite Image.

¿Another geometry?

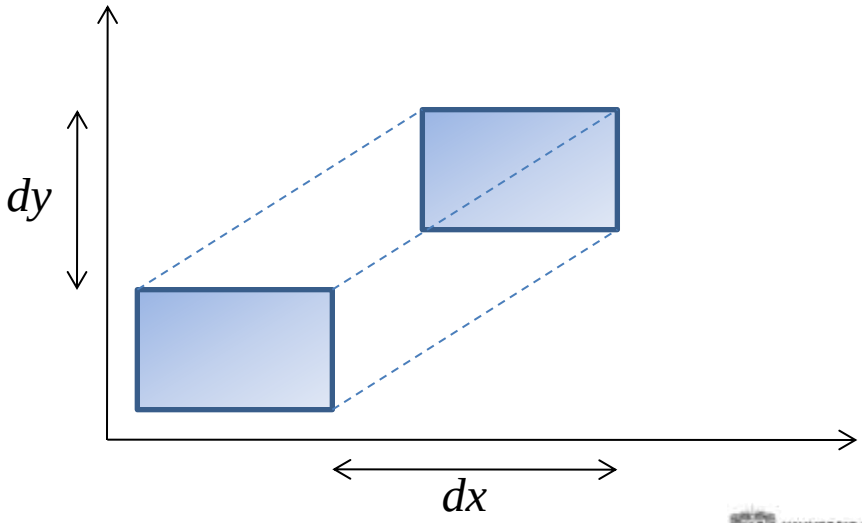
- The primary forms are not directly constructed, like straight lines, circles, triangles, etc.
- … They are more like a set of procedures (algorithms) to rotate, shift, re-scale and/or distort an original shape.

Affine transformations in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

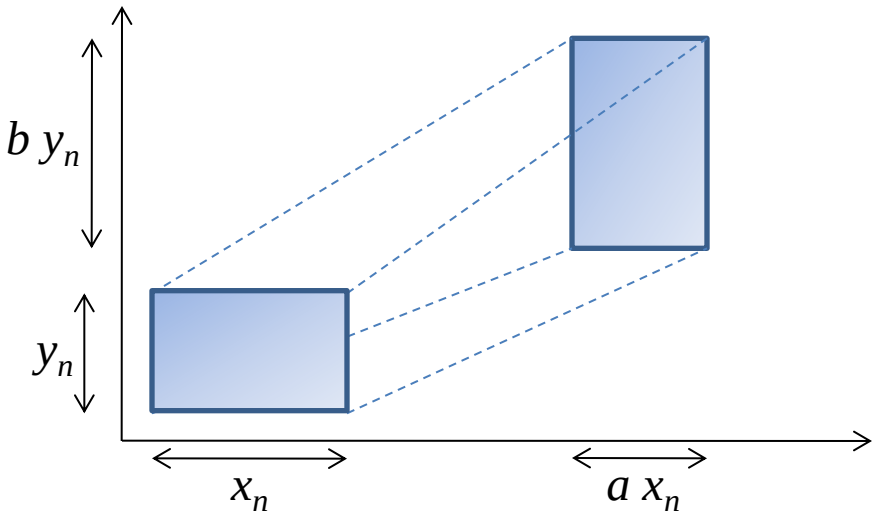
↓ Rotate Scale distort

↓ displace



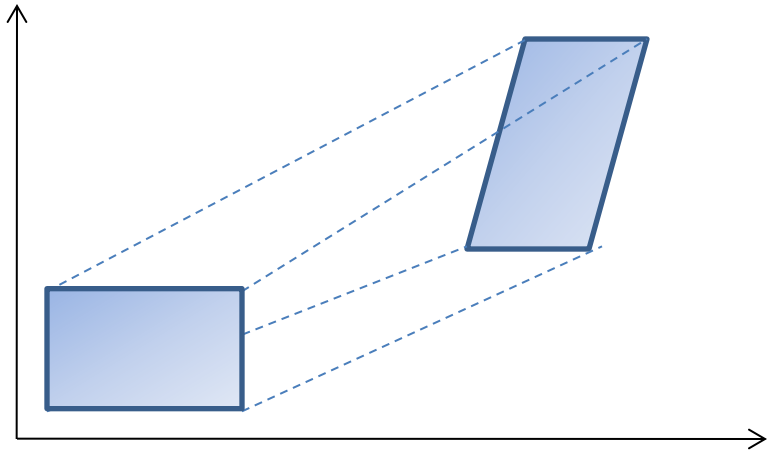
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

A simple translation by (dx,dy)



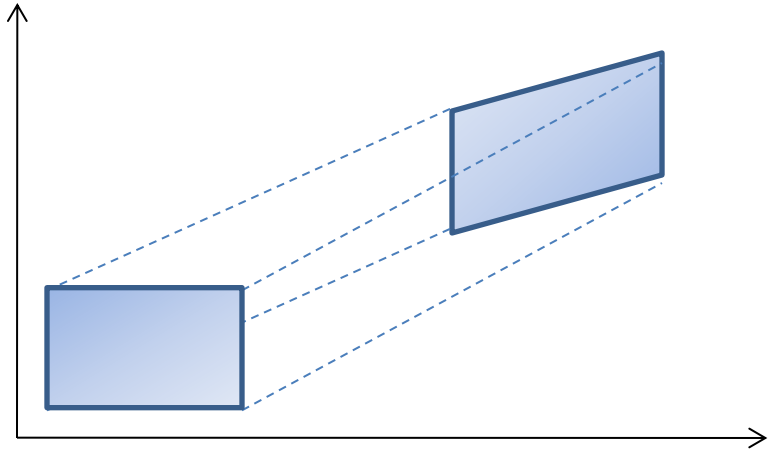
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

A simple resizing by (a,b), with translation -if $ab=1$, preserves the area-



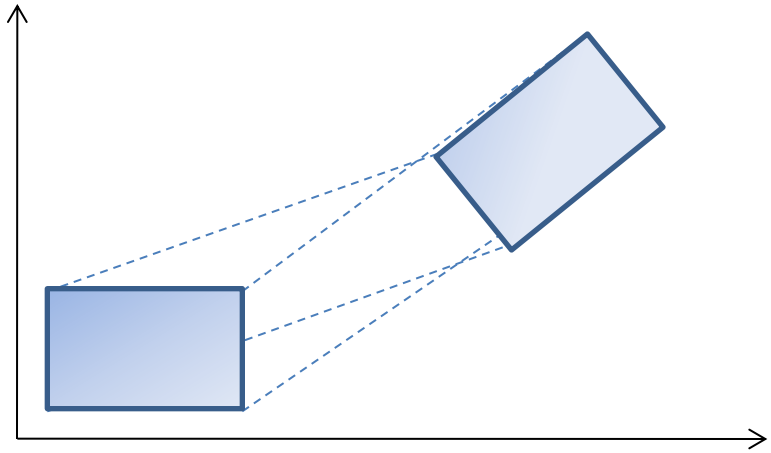
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Shearing parallel to x



$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Shearing parallel to y

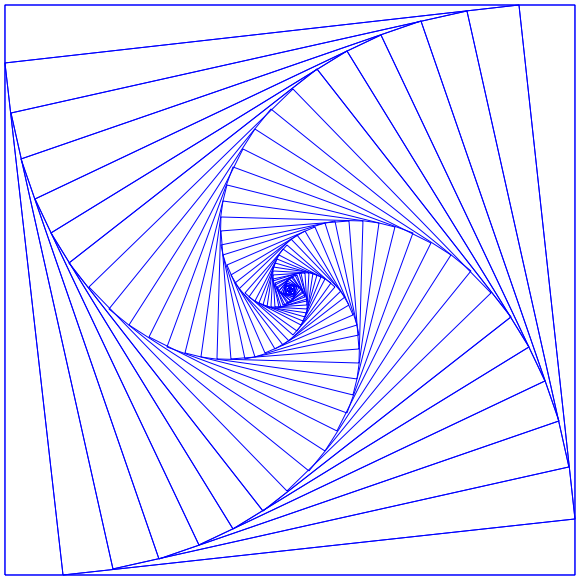


$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Rotation by an angle θ

We can iterate a simple affine transformation:

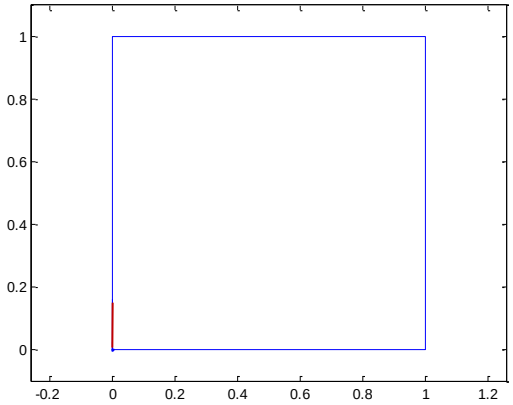
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.0 \end{bmatrix}$$



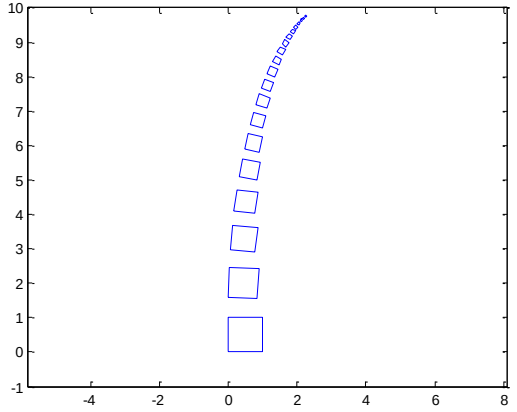
Iterated Function System

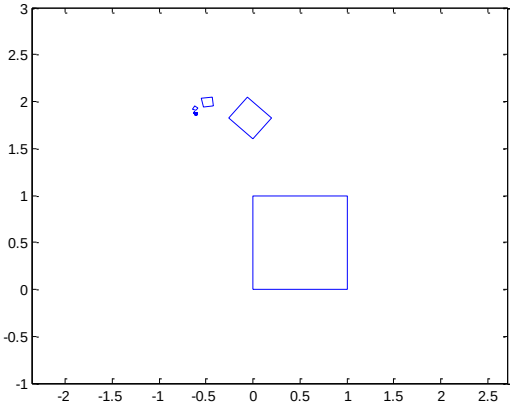
```
xy    = [0 0; 1 0; 1 1; 0 1; 0 0]';
A     = [0.9 -0.1; 0.1 0.9];
b     = repmat([0.1; 0.0],1,5);
for i = 1:100
    plot(xy(1,:),xy(2,:)); hold on
    xy = A*xy + b;
end
```

Four interesting (¿?) affine transformations in $\mathbb{R}^2$



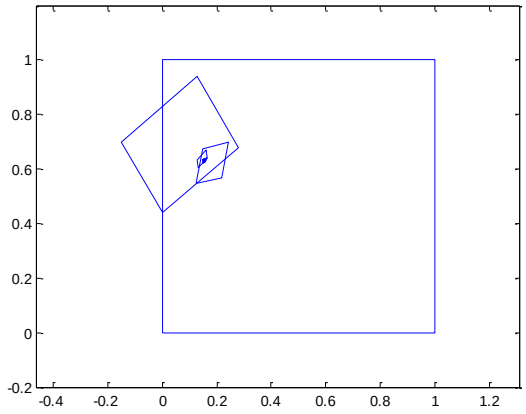
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$





$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$



$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$

Nature's geometry

Starting with **x** = (0.5, 0.5):

plot(**x**)

with prob. 0.01, $\mathbf{x} \leftarrow A_1 \mathbf{x} + b_1$;

with prob. 0.85, $\mathbf{x} \leftarrow A_2 \mathbf{x} + b_2$;

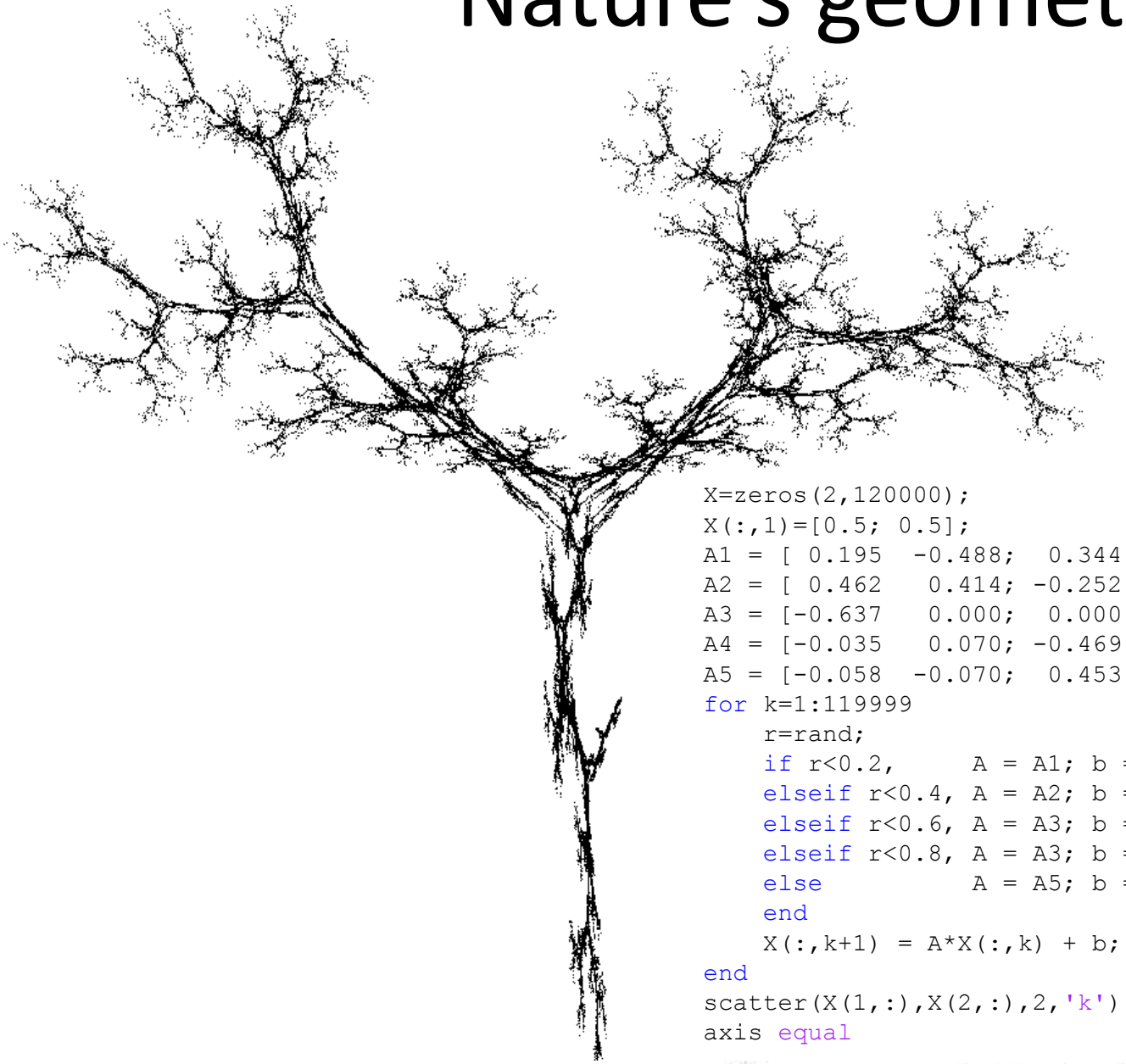
with prob. 0.07, $\mathbf{x} \leftarrow A_3 \mathbf{x} + b_3$;

with prob. 0.07, $\mathbf{x} \leftarrow A_4 \mathbf{x} + b_4$;

Repeat until we have enough dots

```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.00  0.00;  0.00  0.16]; b1 = [0.0; 0.00];
A2 = [ 0.85  0.04; -0.04  0.85]; b2 = [0.0; 1.60];
A3 = [ 0.20 -0.26;  0.23  0.22]; b3 = [0.0; 1.60];
A4 = [-0.15  0.28;  0.26  0.24]; b4 = [0.0; 0.44];
for k=1:119999
    r=rand;
    if     r<.01, A = A1; b = b1;
    elseif r<.86, A = A2; b = b2;
    elseif r<.93, A = A3; b = b3;
    else          A = A4; b = b4;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'g')
axis equal
```

Nature's geometry



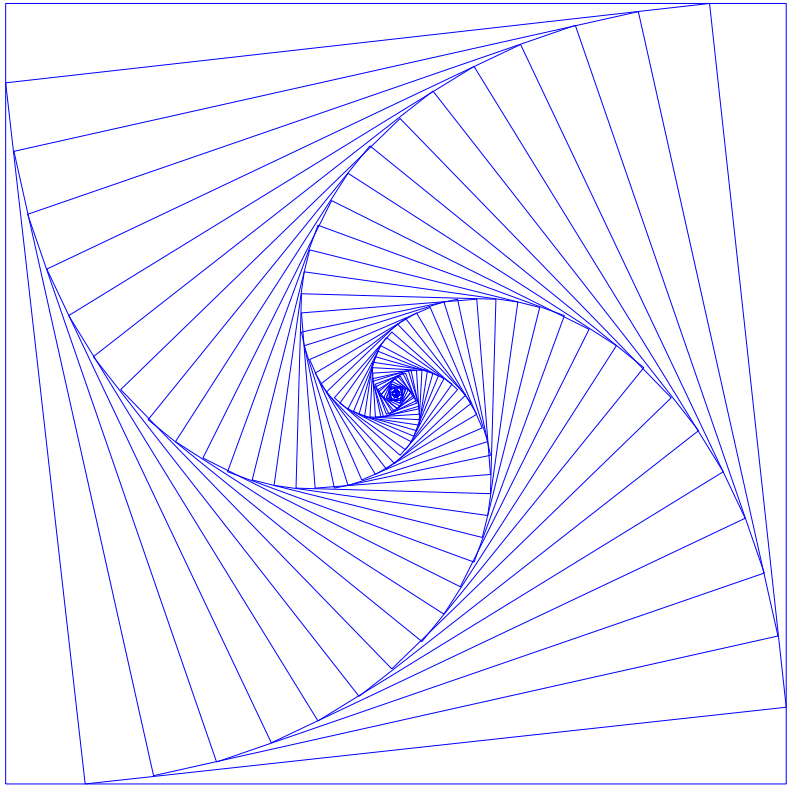
```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.195  -0.488;  0.344  0.443]; b1 = [0.4431; 0.2452];
A2 = [ 0.462   0.414; -0.252  0.361]; b2 = [0.2511; 0.5692];
A3 = [-0.637   0.000;  0.000  0.501]; b3 = [0.8562; 0.2512];
A4 = [-0.035   0.070; -0.469  0.022]; b4 = [0.4884; 0.5069];
A5 = [-0.058  -0.070;  0.453 -0.111]; b5 = [0.5976; 0.0969];
for k=1:119999
    r=rand;
    if r<0.2,      A = A1; b = b1;
    elseif r<0.4, A = A2; b = b2;
    elseif r<0.6, A = A3; b = b3;
    elseif r<0.8, A = A3; b = b3;
    else           A = A5; b = b5;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'k')
axis equal
```

```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.387  0.430;  0.430 -0.387]; b1 = [0.2560; 0.5220];
A2 = [ 0.441 -0.091; -0.009 -0.322]; b2 = [0.4219; 0.5059];
A3 = [-0.468  0.020; -0.113  0.015]; b3 = [0.4000; 0.4000];
for k=1:119999
    r=rand;
    if r<.333,      A = A1; b = b1;
    elseif r<.666, A = A2; b = b2;
    else            A = A3; b = b3;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'g')
```

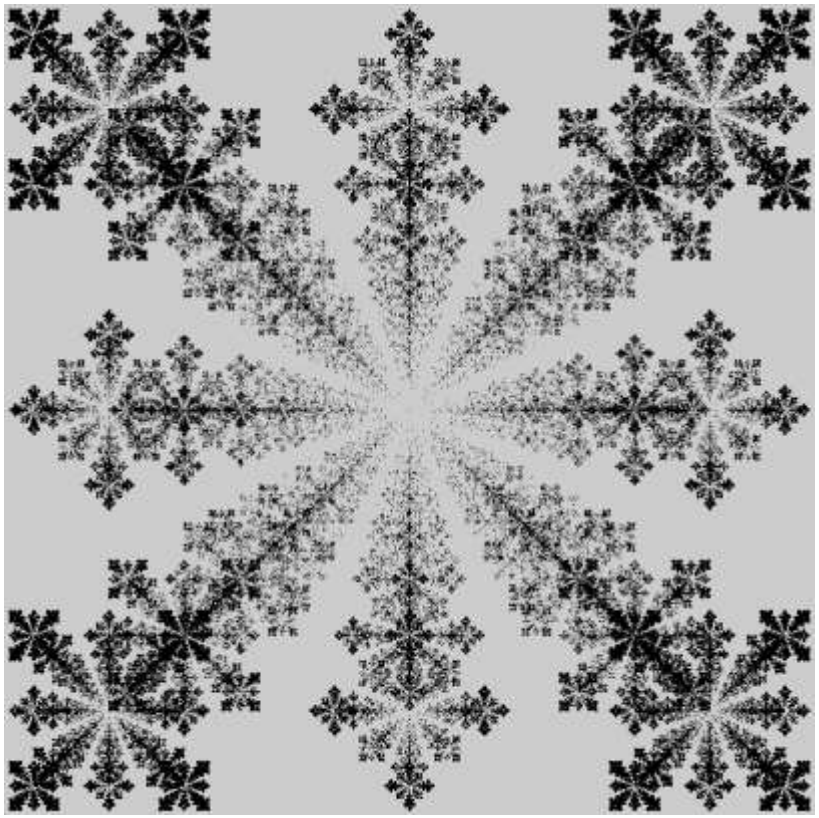
```
X=zeros(2,120000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.50  0.00;  0.00  0.75]; b1 = [0.25; 0.00];
A2 = [ 0.25 -0.20;  0.10  0.30]; b2 = [0.25; 0.50];
A3 = [ 0.25  0.20; -0.10  0.30]; b3 = [0.50; 0.40];
A4 = [ 0.20  0.00;  0.00  0.30]; b4 = [0.40; 0.55];
for k=1:119999
    r=rand;
    if r<0.25,      A = A1; b = b1;
    elseif r<0.50, A = A2; b = b2;
    elseif r<0.75, A = A3; b = b3;
    else            A = A4; b = b4;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'g')
axis equal
```

Homework # 3

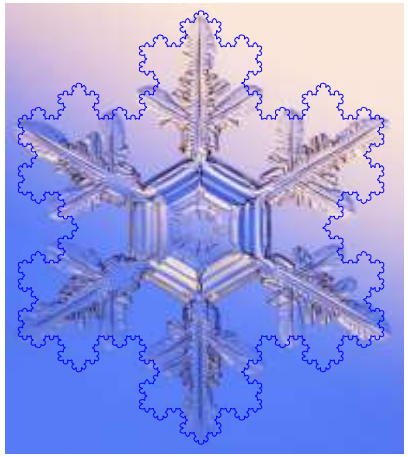
A. Spend no more than one hour of your time trying to modify the self-affine IFS of the left to generate the best romanescu broccoli you can ☺

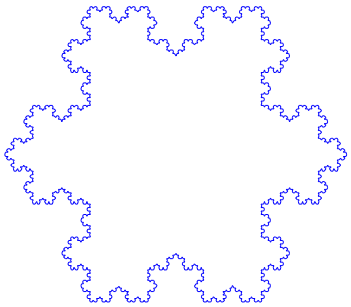


Not only plants

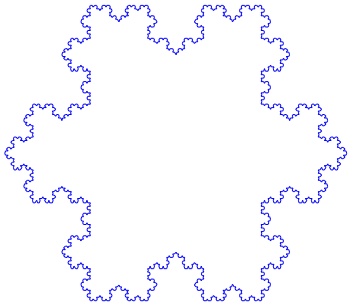


```
X=zeros(2,250000);
X(:,1)=[0.5; 0.5];
A1 = [ 0.75  0.00;  0.00  0.75]; b1 = [0.125; 0.125];
A2 = [ 0.50 -0.50;  0.50  0.50]; b2 = [0.500; 0.000];
A3 = [ 0.25  0.00;  0.00  0.25]; b3 = [0.000; 0.750];
A4 = [ 0.25  0.00;  0.00  0.25]; b4 = [0.750; 0.750];
A5 = [ 0.25  0.00;  0.00  0.25]; b5 = [0.000; 0.000];
A6 = [ 0.25  0.00;  0.00  0.25]; b6 = [0.750; 0.000];
for k=1:249999
    r=rand;
    if r<0.1667,     A = A1; b = b1;
    elseif r<0.3333, A = A2; b = b2;
    elseif r<0.5,    A = A3; b = b3;
    elseif r<0.6667, A = A4; b = b4;
    elseif r<0.8333, A = A5; b = b5;
    else             A = A6; b = b6;
    end
    X(:,k+1) = A*X(:,k) + b;
end
scatter(X(1,:),X(2,:),2,'k')
axis equal
```





¿



?

L-Systems

Aristid Lindenmayer, 1925 - 1989

THE ALGORITHMIC BEAUTY OF PLANTS

PRZEMYSLAW PRUSINKIEWICZ • ARISTID LINDENMAYER

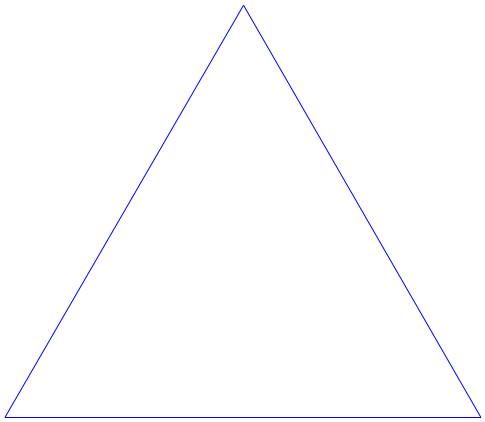
In many growth processes of living organisms, especially of plants, regularly repeated appearances of certain multicellular structures are readily noticeable.... In the case of a compound leaf, for instance, some of the lobes (or leaflets), which are parts of a leaf at an advanced stage, have the same shape as the whole leaf has at an earlier stage…. Organic form itself is found, mathematically speaking, to be a function of time.... We might call the form of an organism an event in space-time, and not merely a configuration in space… The idea of the form implicitly contains also the history of such a form.

L-Systems: Formal grammar and rewriting

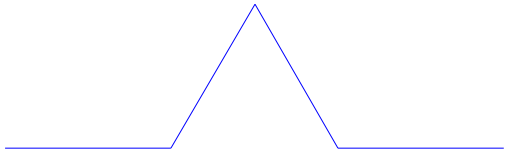
- Chomsky's formal grammar: Rewriting to describe the syntactic features of natural languages (1950's)
- Backus and Naur: Rewriting-based notation provides a formal definition of computer programming languages (algol, 1960's)
- The recognition of the equivalence started a period of fascination with syntax, grammars and their application to computer science (1970's)

L-Systems: Rewriting

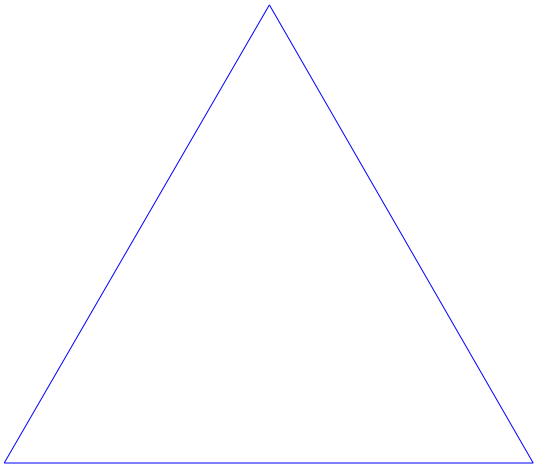
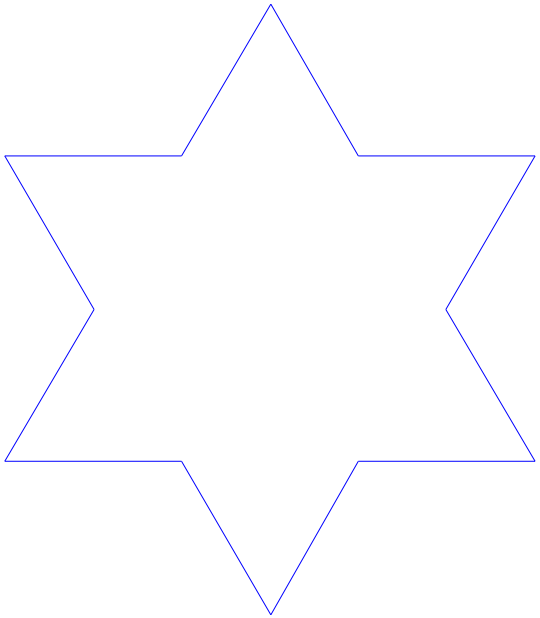
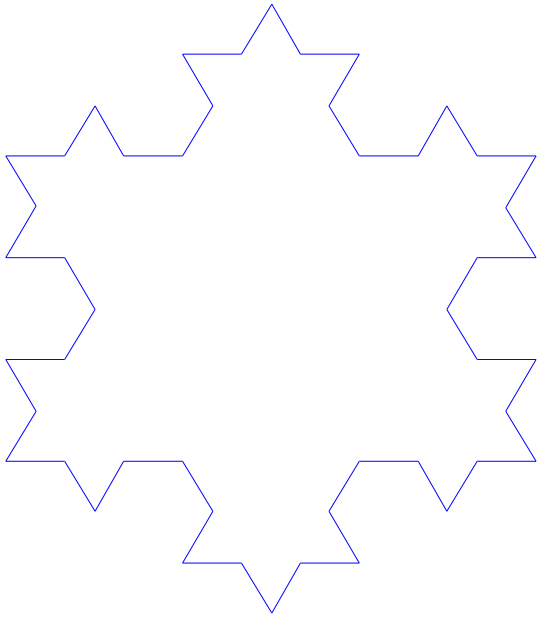
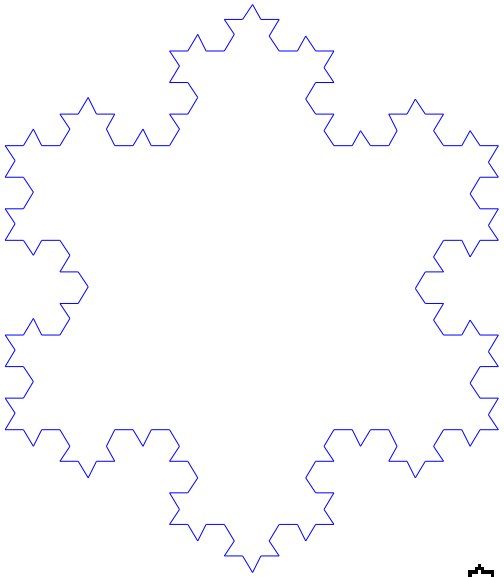
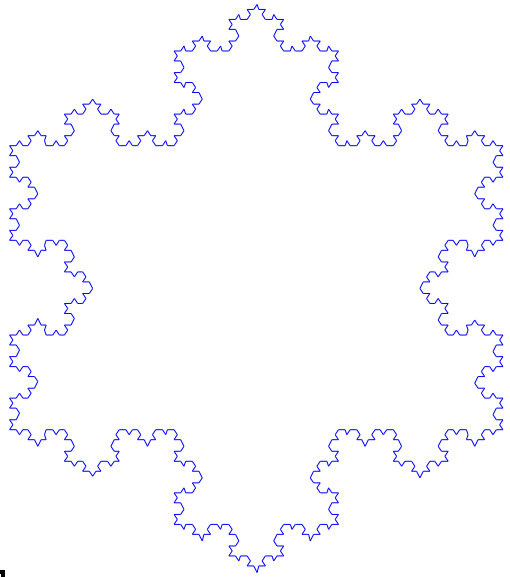
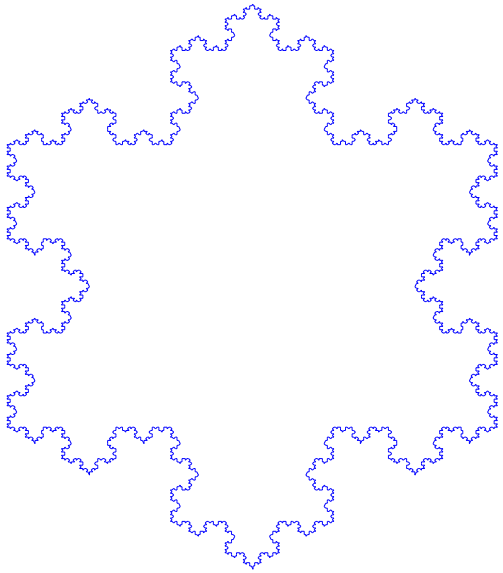
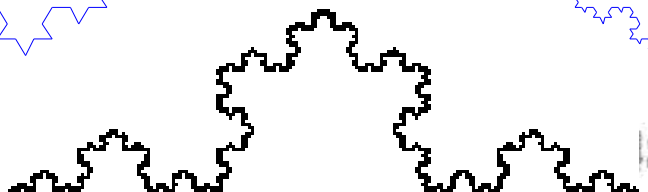
- Rewriting: define complex objects by successively replacing parts of a simple initial object using a set of rewriting rules.
- The classic example is the snowflake curve proposed in 1905 by von Koch, which Mandelbrot restates as follows:
  - One begins with two shapes, an initiator and a generator. The latter is an oriented broken line made up of N equal sides of length r. Thus each stage of the construction begins with a broken line and consists in replacing each straight interval with a copy of the generator, reduced and displaced so as to have the same end points as those of the interval being replaced.



Initiator



Generator

L-Systems: Rewriting

- A tree grows from a seed : How can new cells be generated from old cells?
- The seed cell is known as an *axiom*
- The instructions of how to grow new cells are known as *production rules*
- For example,

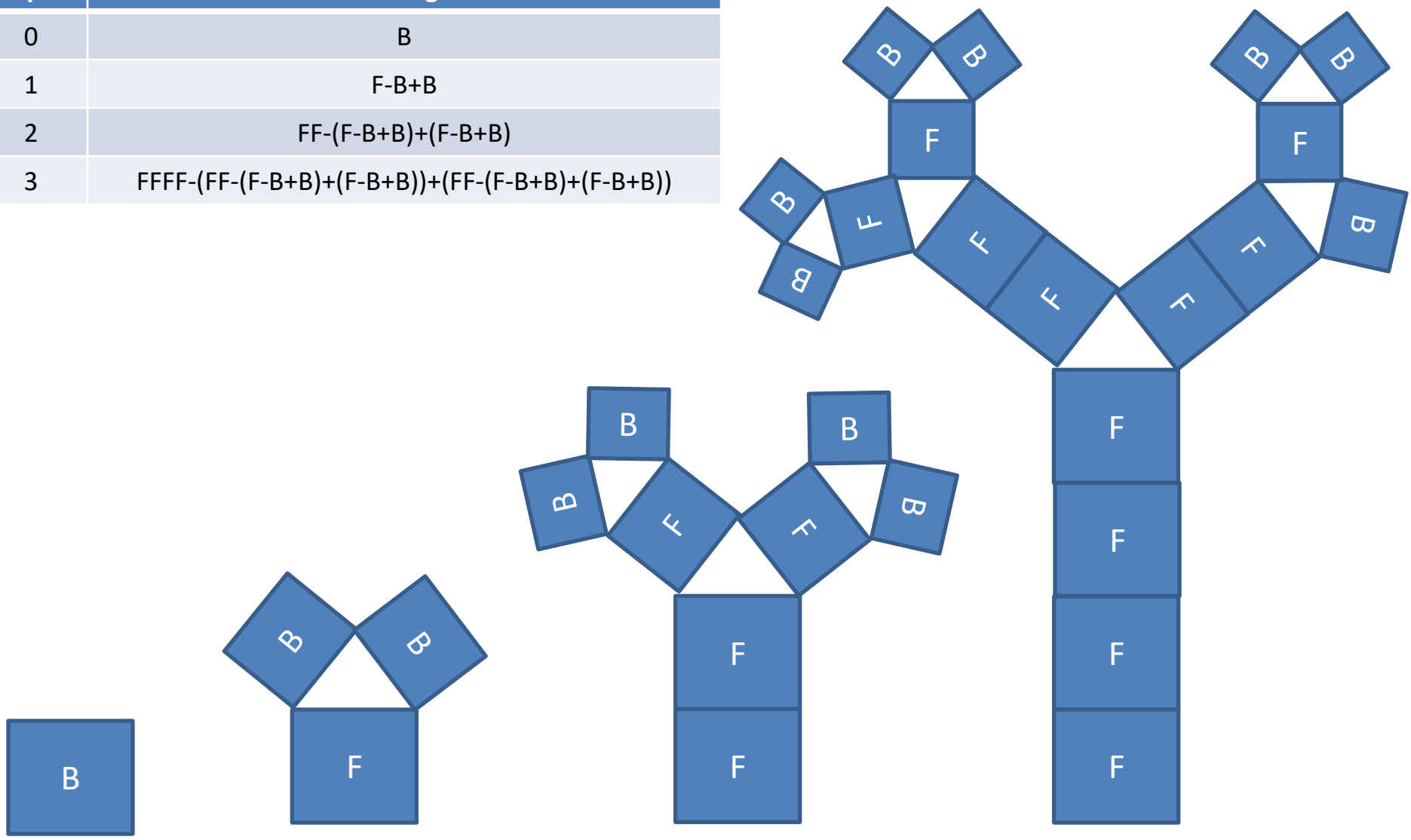
Axiom : B

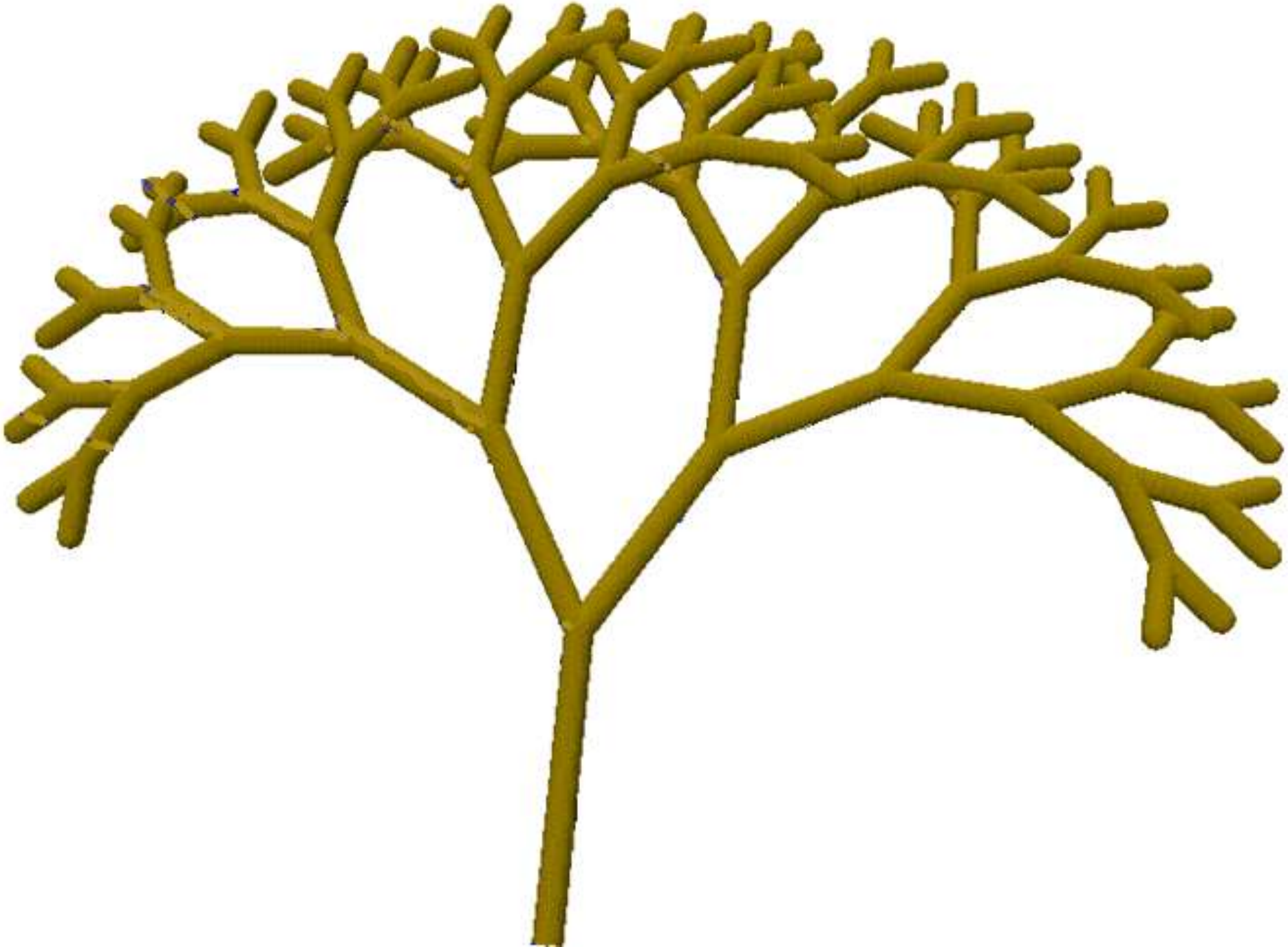
Rules : B$\rightarrow$F-B+B, F$\rightarrow$FF

Depth	String
0	B
1	F-B+B
2	FF-(F-B+B)+(F-B+B)
3	FFFF-(FF-(F-B+B)+(F-B+B))+(FF-(F-B+B)+(F-B+B))

L-Systems: Rewriting

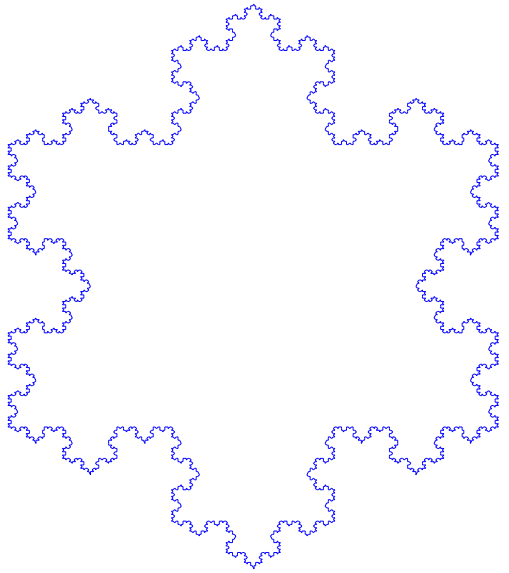
Depth	String
0	B
1	F-B+B
2	FF-(F-B+B)+(F-B+B)
3	FFFF-(FF-(F-B+B)+(F-B+B))+(FF-(F-B+B)+(F-B+B))





Quasi-L System

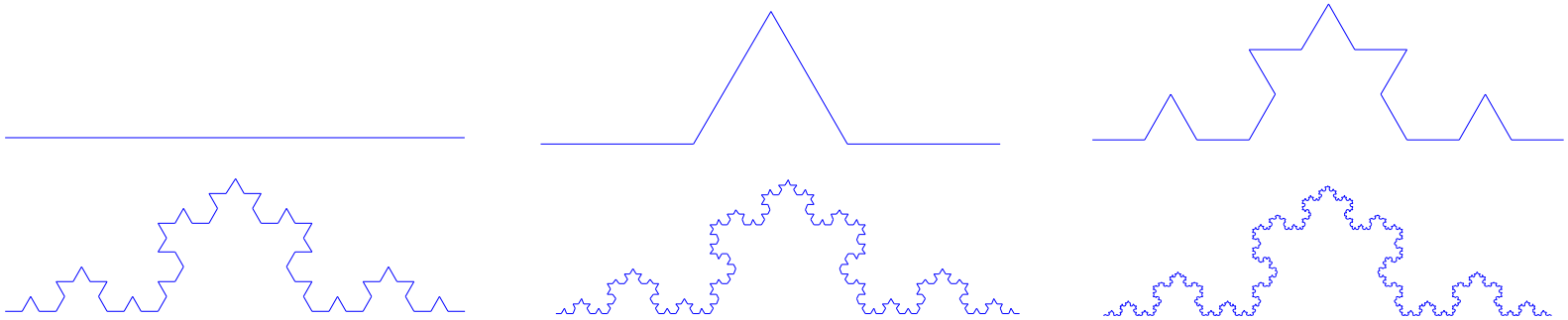
```
function Tortuga(delta,Axioma)
    x0 = 0; y0 = 0; distancia = 10; angulo = 0;
    i = 1;
    clf
    while Axioma(i)~='.'
        switch Axioma(i)
            case 'f'
                x1 = x0 + distancia*cos(angulo);
                y1 = y0 + distancia*sin(angulo);
                plot([x0 x1],[y0 y1])
                hold on
                x0 = x1; y0 = y1;
            case 'l'
                angulo = angulo + delta;
            case 'r'
                angulo = angulo - delta;
        end
        i = i+1;
    end
```



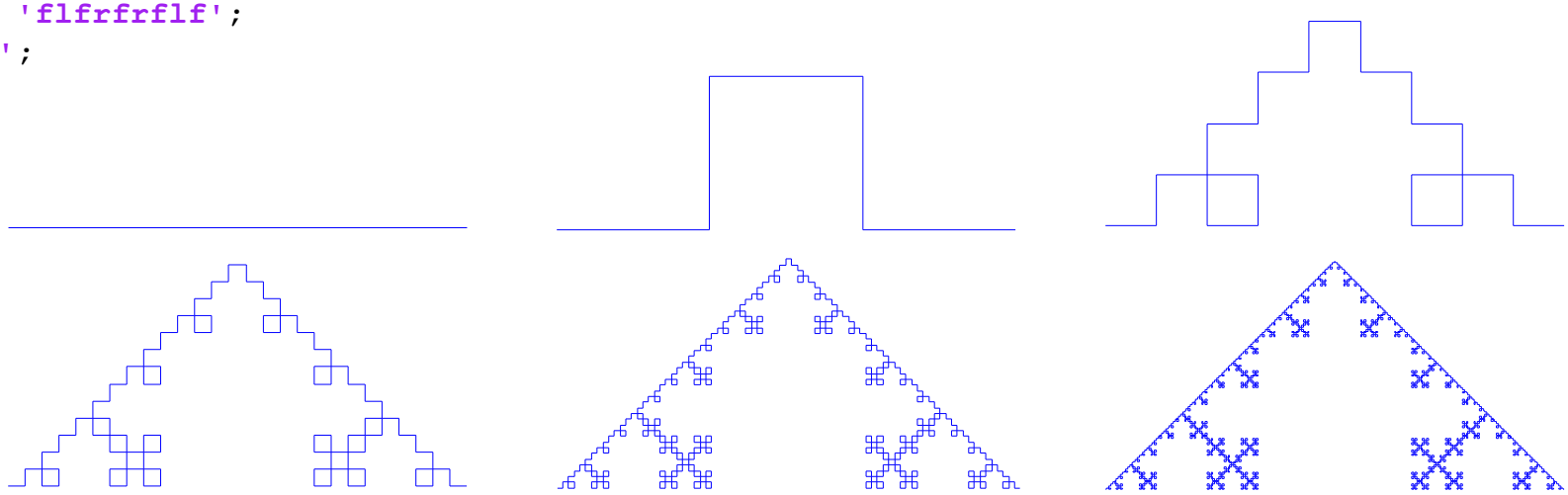
```
function SistemaCasiL(delta,Generador,Axioma,n)
    while n>1
        buffer = '';
        for i=1:length(Axioma)
            if Axioma(i) == 'f'
                buffer = [buffer Generador];
            else
                buffer = [buffer Axioma(i)];
            end
        end
        Axioma = buffer;
        n = n-1;
    end
    Axioma = [Axioma '.'];
    Tortuga(delta,Axioma)
```

```
function secuencia(delta,Generador,Axioma)
    for n=1:5
        SistemaCasiL(delta,Generador,Axioma,n);
        drawnow
        pause(1)
    end
```

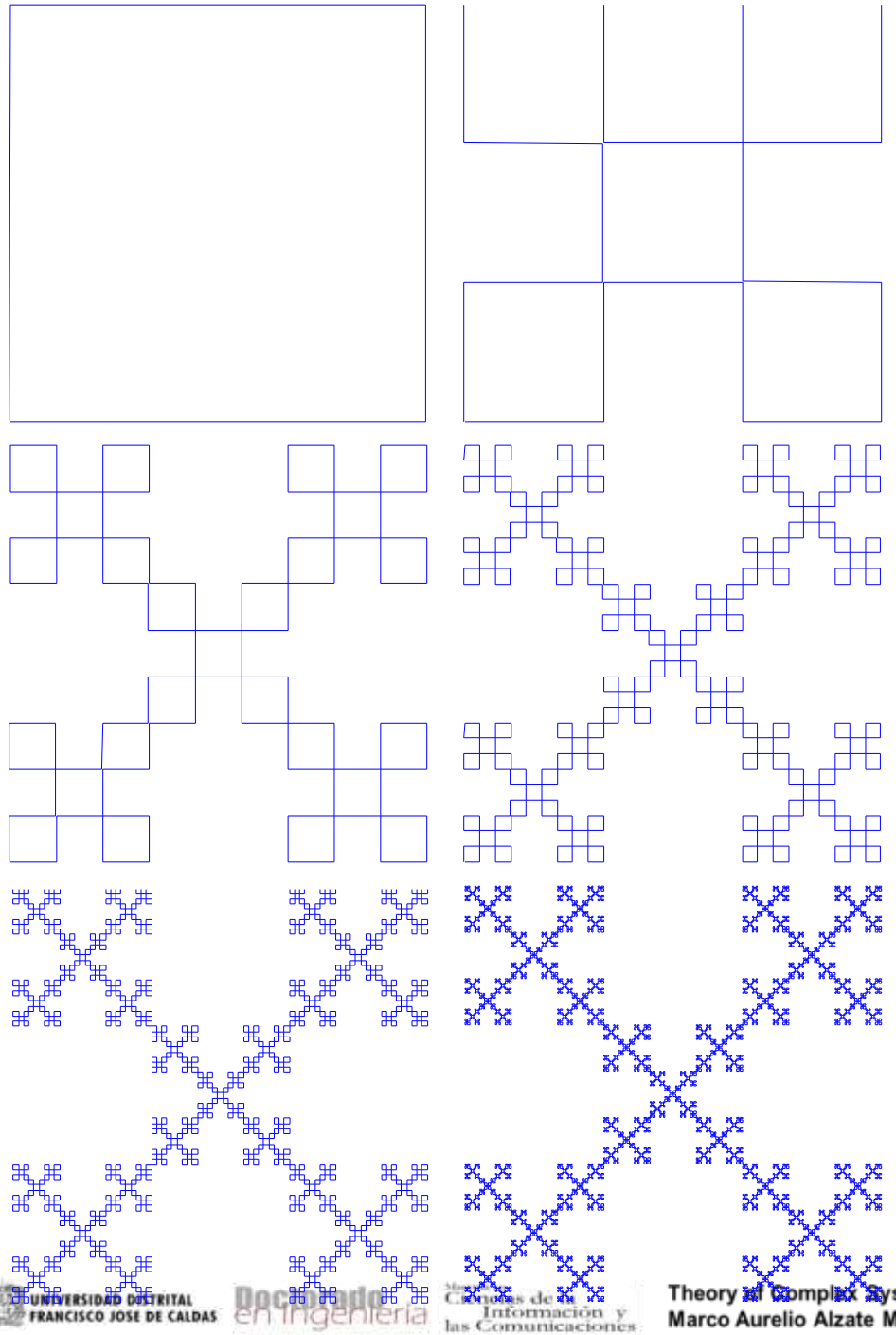
```
% Copo de nieve de Von Koch
delta = pi/3;
Generador = 'flfrrflf';
Axioma = 'f';
```



```
% Crecimiento de moho
delta = pi/2;
Generador = 'flfrfrflf';
Axioma = 'f';
```



```
% Otro crecimiento cristalino
delta = pi/2;
Generador = 'flfrfrflf';
Axioma = 'flflflfl';
```



This is everything about mathematics

$A_1, A_2, ..., A_k$ - axioms and previously proved theorems
Formal proof of a sentence P is a sequence of statements

$$S_1, S_2, ..., S_n$$

where:

1. S_n is P and one of the following holds:
2a. S_i is one of $A_1, A_2, ..., A_k$
or
2b. S_i follows from the previous statements by a valid argument using the rules of reasoning (grammar rules)

Math is simply a (formal) language

Example: Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space.

Axioms: **(1)** $\mathbf{P}(\Omega) = 1$

(2) If $A \in \mathscr{F}$, $\mathbf{P}(A) \geq 0$

(3) If $A, B \in \mathscr{F}$ and $A \cap B = \phi$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

Theorem: If $A \subseteq B$, $\mathbf{P}(A) \leq \mathbf{P}(B)$.

Proof: $B = A \cup (B \cap A^C)$

$\mathbf{P}(B) = \mathbf{P}(A) + \mathbf{P}(B \cap A^C)$

$\mathbf{P}(B \cap A^C) \geq 0$

$\mathbf{P}(B) \geq \mathbf{P}(A)$.

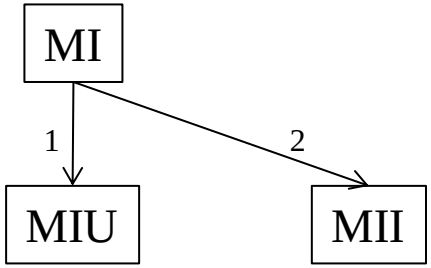
Homework # 3

B. A formal system consists of the symbols M, I and U, along with some grammar rules to contruct strings from previously known ones.

Axiom: MI
Grammar rules: 1. If *x*I, then *x*IU
2. If M*x*, then M*xx*
3. if *x*III*y* then *x*U*y*
4. if *x*UU*y* then *xy*
Homework: Prove MU, if it is possible

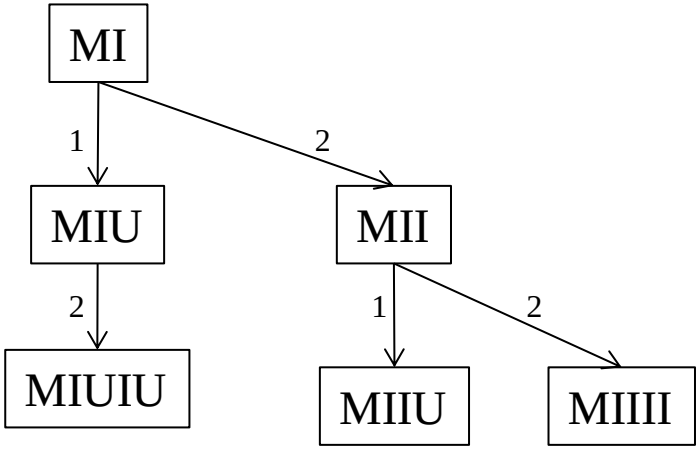
(where *x* and *y* is any sequence –even an empty sequence – of symbols M, I and/or U)

Douglas R. Hofstadter,
"Gödel, Escher, Bach: an Eternel Golden Braid",
Basic Books, 1979

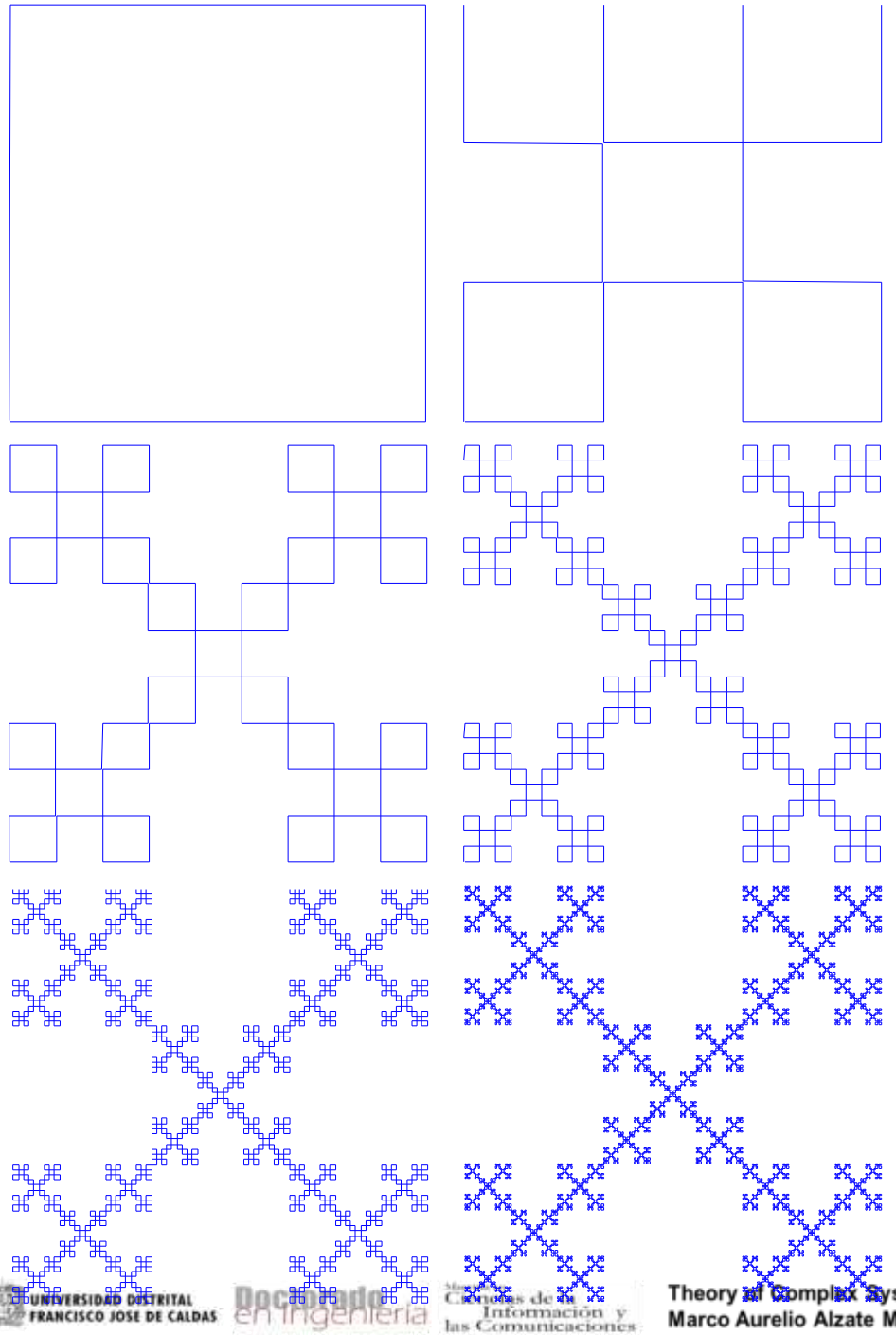


1. If *x*I, then *x*IU
2. If M*x*, then M*xx*
3. if *x*III*y* then *x*U*y*
4. if *x*UU*y* then *xy*

1. If *x*I, then *x*IU
2. If M*x*, then M*xx*
3. if *x*III*y* then *x*U*y*
4. if *x*UU*y* then *xy*

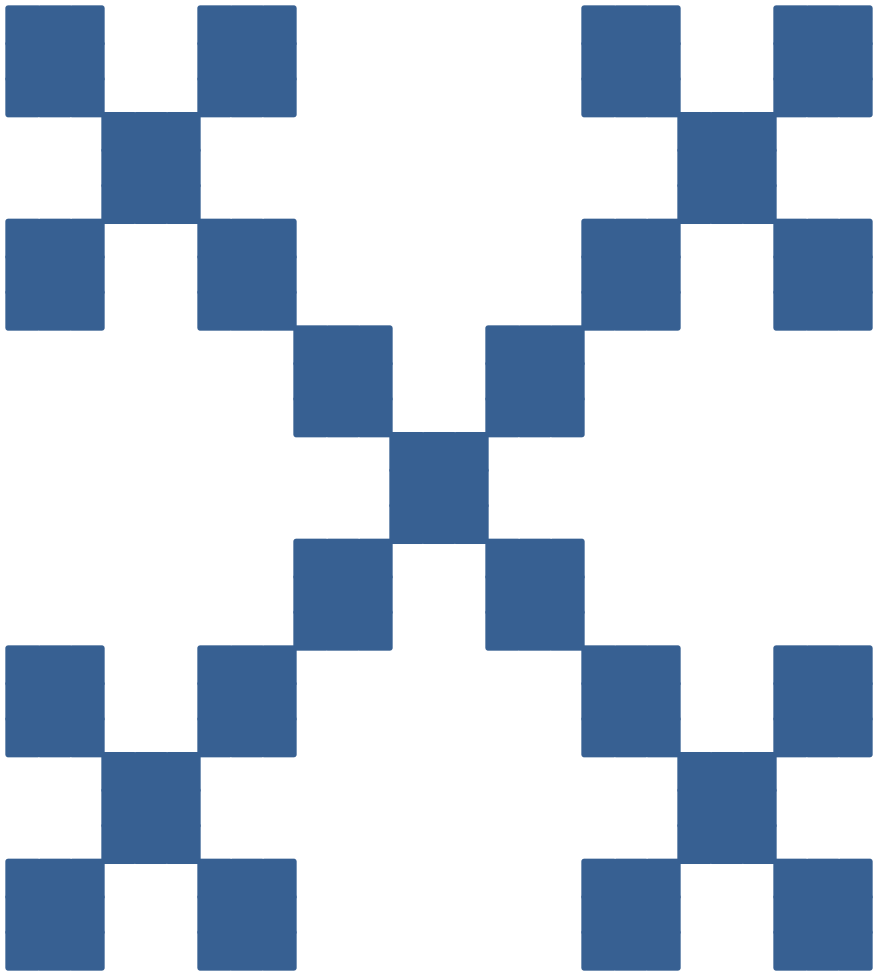


```
% Otro crecimiento cristalino
delta = pi/2;
Generador = 'flfrfrflf';
Axioma = 'flflflfl';
```

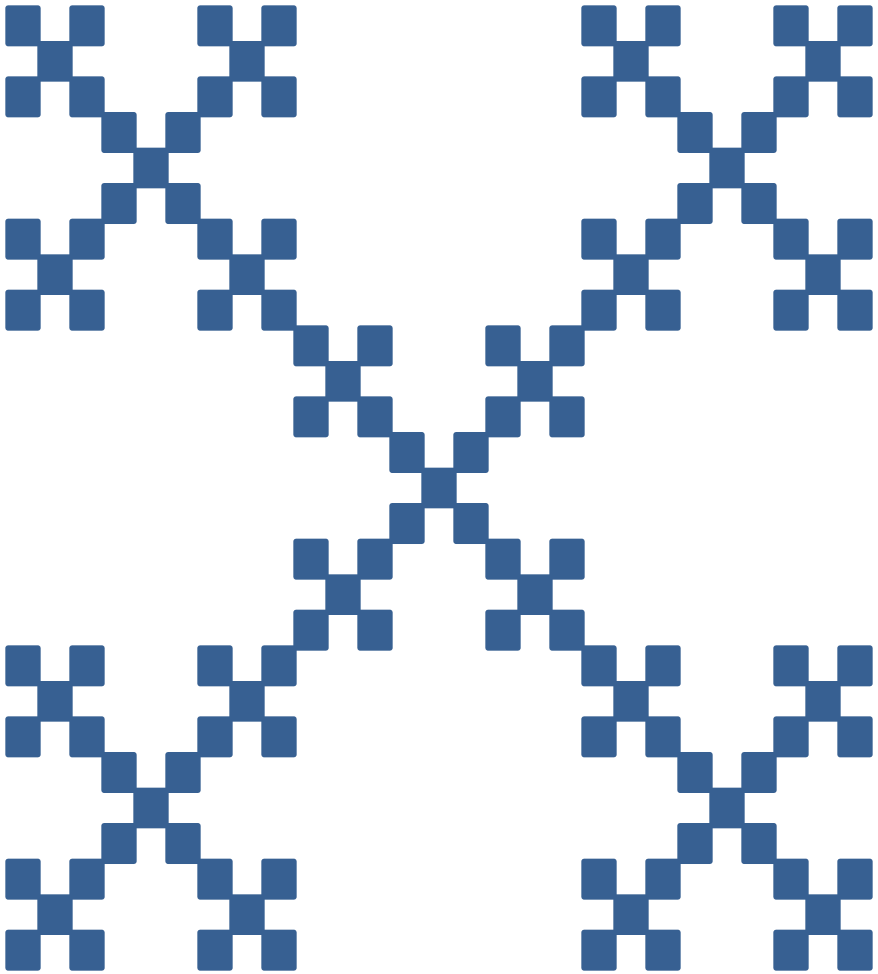


Yet another algorithm to construct the same fractal

Yet another algorithm to construct the same fractal

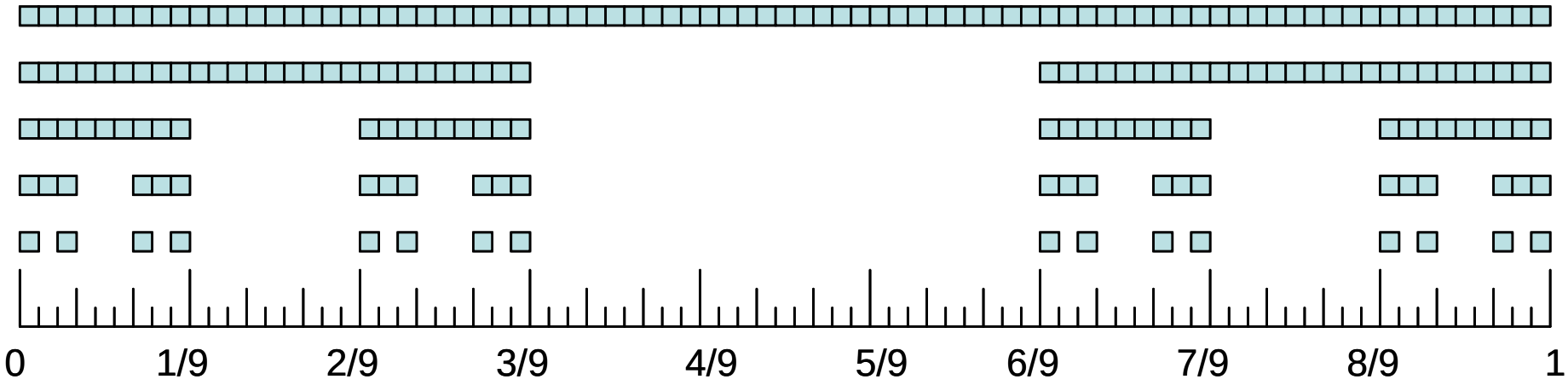


Yet another algorithm to construct the same fractal



Yet another algorithm to construct the same fractal

The Cantor Set



How many intervals are there in the Cantor set?

At the n^{th} step, we remove 2^n intervals, each of length $3^{-(n+1)}$

Total length of removed intervals:

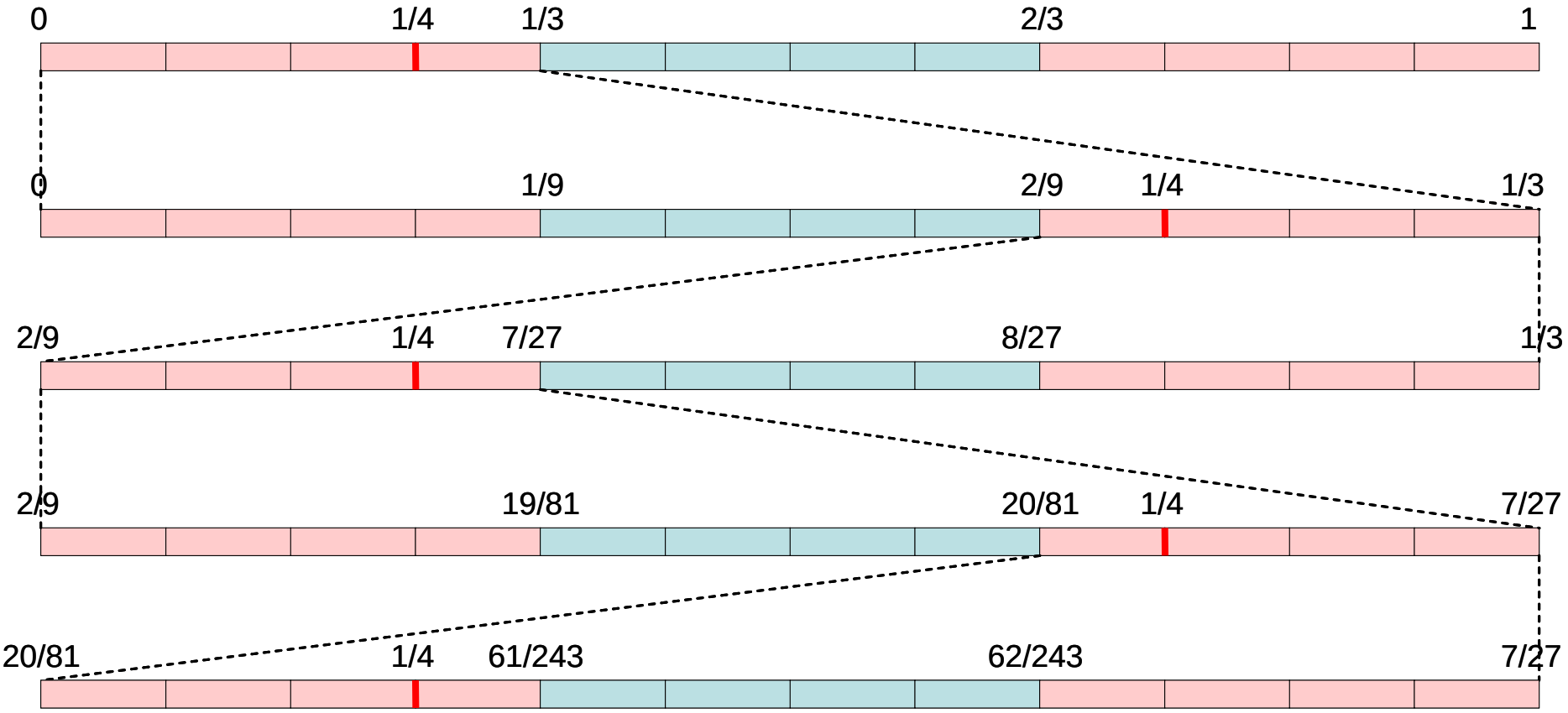
$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

Can we count the number of isolated points in the Cantor set?

Yes, they are the limits of the intervals…. Sure?

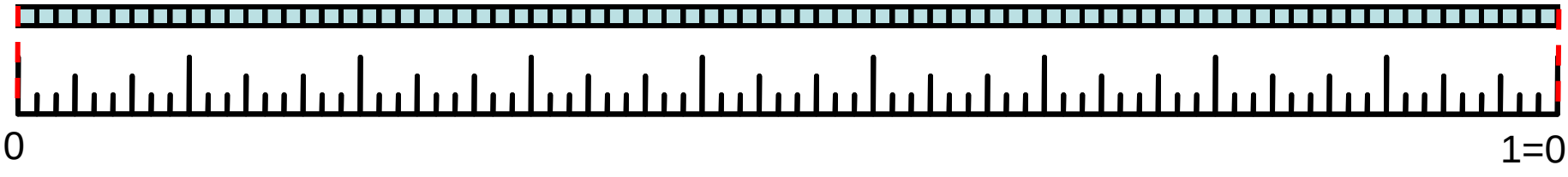
The Cantor Set

Those isolated points are not only the end points of middle third intervals. For instance, many points (like ¼) are never removed from the Cantor set, despite they are not the end points of middle third intervals:



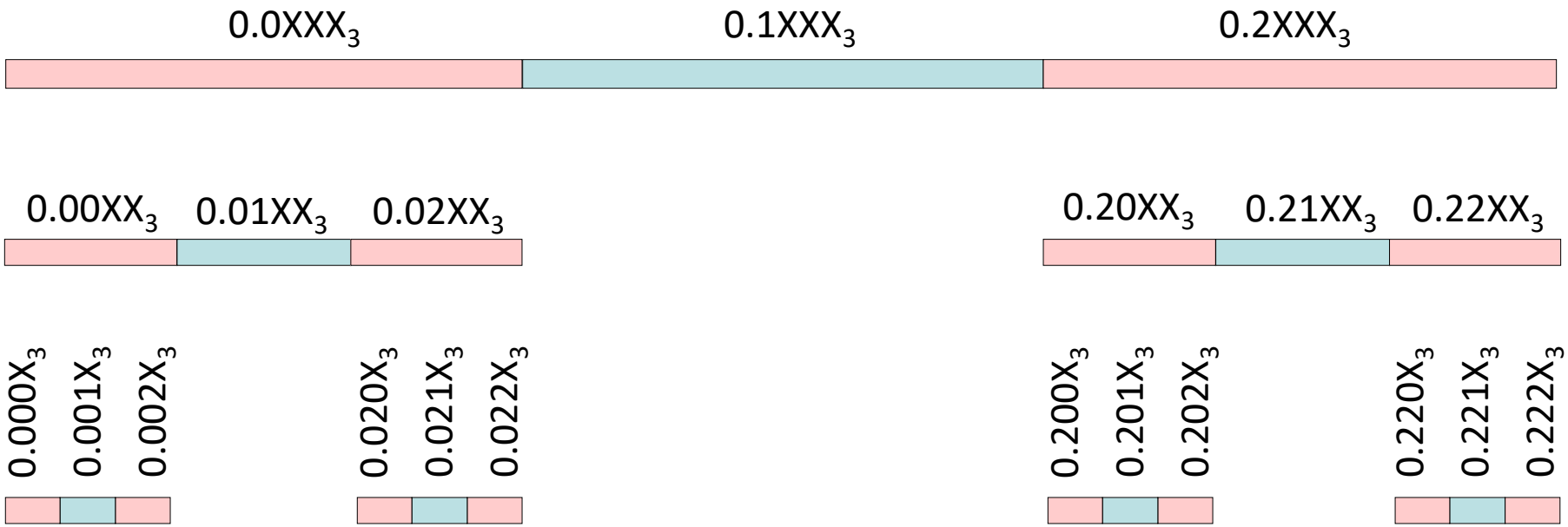
The Cantor Set

Indeed, let us represent the end points in base 3:



$$0.\bar{2}_3 = \sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{2}{3}\left(\frac{1}{1-\frac{1}{3}}\right) = 1$$

Which points do belong to the Cantor set?

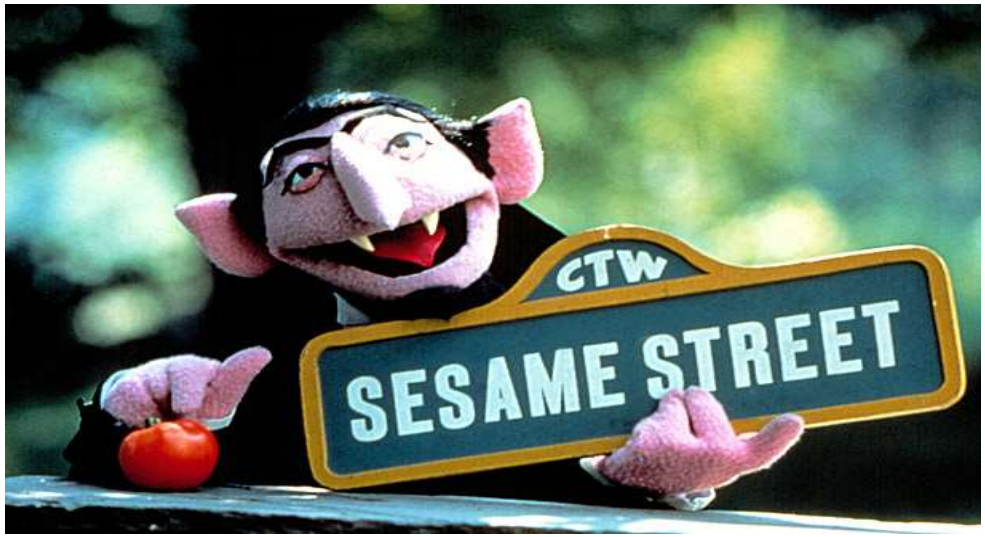


The Cantor set consists of those points $c \in [0, 1]$ that have no 1's in their base-3 expansion

Not only the end points of middle third intervals: $0.\overline{02}_3 = 2\sum_{n=1}^{\infty}\frac{1}{3^{2n}} = \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{2}{9}\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{1}{4}$

How many points are there in the Cantor set?

To answer this question it is interesting to learn how to count:



- Two sets X and Y are said to have the same *cardinality* (number of elements) if there is an invertible mapping that pairs each element $x \in X$ with precisely one $y \in Y$: one-to-one correspondence.
- If a set X can be put into one-to-one correspondence with the natural numbers, $\mathbb{N}$ = {1,2,3,…}, X is said to be countable.
- For instance, the even natural numbers, $\mathbb{E}$ = {2,4,6,…}, is countable through the invertible mapping $f(n) = 2n$. Hence, there are exactly as many even numbers as natural numbers, although we might think there would be only half as many, because odd numbers are missing. ☺

How many elements are there in countable sets?

- Equivalently, a set X is said to be countable if it can be written as a list $X = \{x_1, x_2, x_3, \dots\}$ with every $x \in X$ appearing somewhere in the list, i.e., given any $x \in X$, there is some finite $n \in \mathbb{N}$ such that $x_n = x$.
- A convenient way to exhibit such a list is to give an algorithm that systematically associates the index n with the element $x_n \in X$.
- For example: Are the integers $\mathbb{Z}$ = {…,-3,-2,-1,0,1,2,3,…} countable?
  - Find a clever way to start: it would not be helpful to go one way from zero and, after finisihing in that direction, come to zero again to begin on the other way: we will never reach the end of the first half.
  - This is a better way to go: $\mathbb{Z} = \{z_1 = 0, z_2 = 1, z_3 = -1, z_4 = 2, z_5 = -2, z_6 = 3, z_7 = -3, \dots\}$
  - Since every particular integer will appear eventually, integers are countable: There are as many integers as naturals, not twice ☺
  - Indeed, the position of the integer z is $n(z) = \begin{cases} 1 & z = 0 \\ 2z & z > 0 \\ 1-2z & z < 0 \end{cases}$
- Show that the positive rational numbers are countable
  - Wrong way to go: $\mathbb{Q}$ = {1/1, 1/2, 1/3, 1/4, 1/5, … }. We will never finish the 1/n's, so numbers like 2/3 will never be counted.
  - Any suggestion?
  - There are as many rational numbers as natural numbers!

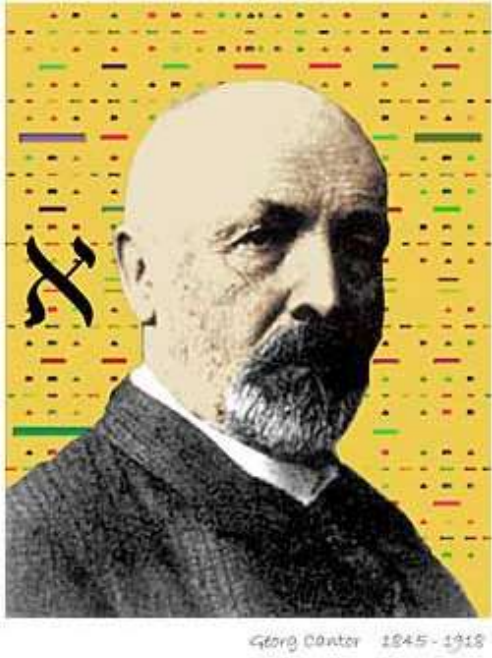
How many real numbers are there?

- Cantor showed that real numbers are uncountable:
  - By contradiction. If they are countable, we could list the numbers in the interval [0, 1] as a set $\{x_1, x_2, x_3, x_4, \ldots\}$.
  - Let us write these numbers in binary form:
    - $x_1 = 0 \,.\, x_{11}x_{12}x_{13}x_{14}\ldots$
    - $x_2 = 0 \,.\, x_{21}x_{22}x_{23}x_{24}\ldots$
    - $x_3 = 0 \,.\, x_{31}x_{32}x_{33}x_{34}\ldots$
    - …

      where x_{ij} is the j^{th} binary digit of the i^{th} real number
  - The contradiction is obtained by finding a number $r \in [0, 1]$ that is not in the list.
    - The first digit of r is $1\text{-}x_{11}$
    - The second digit $1\text{-}x_{22}$
    - …
    - The i^{th} digit of r is $1\text{-}x_{ii}$.
  - Then r is not in the list because it differs from each number in the list at least by one binary place.
- The real numbers are not countable: There are MORE real numbers than natural numbers! Some infinities are bigger than others!

Cantor´s Transinfinities

- $\aleph_0$ is the smallest infinite: the number of natural numbers
  - Hilbert´s Hotel
  - $\aleph_0 = a + \aleph_0 = \aleph_0 + \aleph_0 = a\aleph_0 = \aleph_0^a$
- $\aleph_1$ is the cardinality of the set of subsets of countable sets, $2^{\aleph_0}$



$$¿\aleph_0, \aleph_1, \cdots, \aleph_n, \cdots?$$

Continuum Hypothesis: $\aleph_1 = 2^{\aleph_0}$

is the smallest cardinality bigger than $\aleph_0$

i.e., there is no set $\Im$ with cardinalitiy

$$\aleph_0 < |\Im| < \aleph_1 = 2^{\aleph_0}$$

How many points are there in the Cantor set?

Why Cantor's mental health became weak…

- The Cantor set consists of those points $c \in [0, 1]$ that have no 1's in their base-3 expansion, so they only have 0's and 2's.
- By changing each 2 in the base-3 expansion of the elements of the Cantor set by a 1, we get all the binary numbers in the interval [0, 1], expressed in binary.
- This is just a one-to-one correspondence between the Cantor set and the real interval [0, 1]!
- We just found a set of ISOLATED POINTS in the interval [0, 1] that have as many elements as the whole CONTINUOUS interval!

  - … Obviously, Cantor went crazy! It was 1900, Mandelbrot was not there and fractals were "mathematical monsters"

    - Fractals present many paradoxes like this

Some properties of the Cantor set

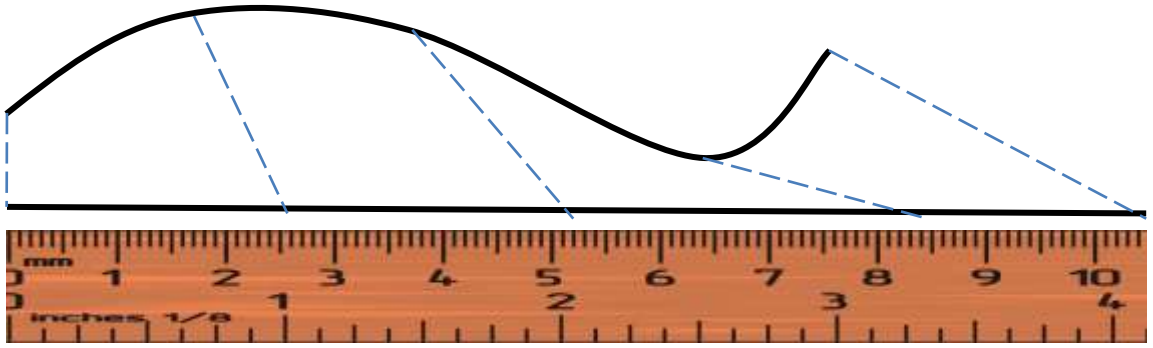
$$\mathcal{C} = \bigcap_{k=0}^{\infty} \bigcup_{i=0}^{(3^k-1)/2} \left[\frac{2i}{3^k}, \frac{2i+1}{3^k} \right] = \left\{ x \in [0,1] : x = \sum_{k=0}^{\infty} a_k 3^{-k},\ a_k \in \{0,2\} \right\}$$

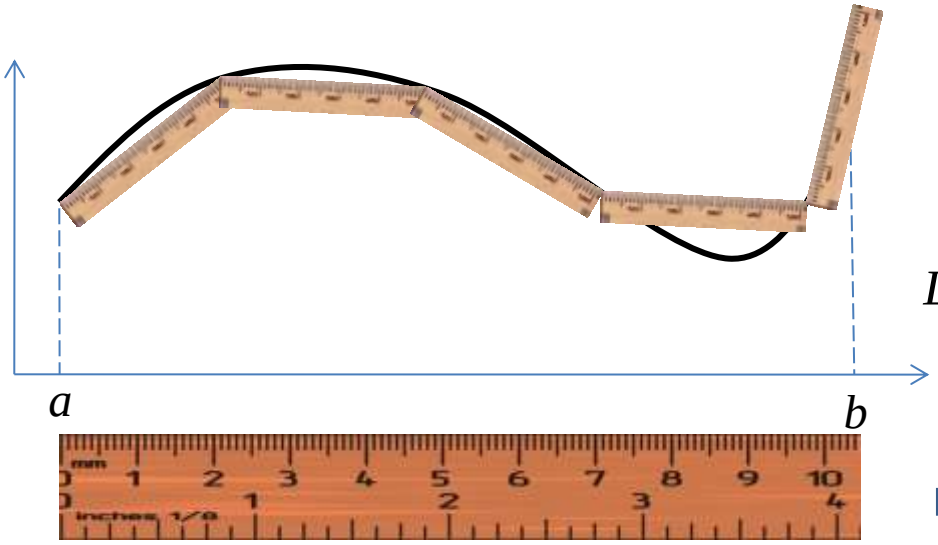
- The Cantor set is self-similar: contains copies of itself at infinitely many scales.

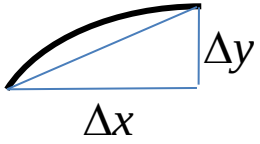
$$\mathcal{C} = \left\{ 3^k x - 2i : x \in \mathcal{C} \cap \left[\frac{2i}{3^k}, \frac{2i+1}{3^k} \right] \right\}_{\substack{k=0,1,2,\ldots \\ i \in \{0,1,2,\ldots,(3^k-1)/2\}}}$$

- The Cantor set has a fine structure: contains detail at arbitrarily small scales.
- Despite its intricate detailed structure, the Cantor set is easily defined through a simple recursive procedure.
- The geometry of the Cantor set is not easily described in classical terms: it is not the set of points that solve any equation, for example.
- Although the Cantor set has an uncountably infinite number of points, its length is zero.

How to measure an arc length





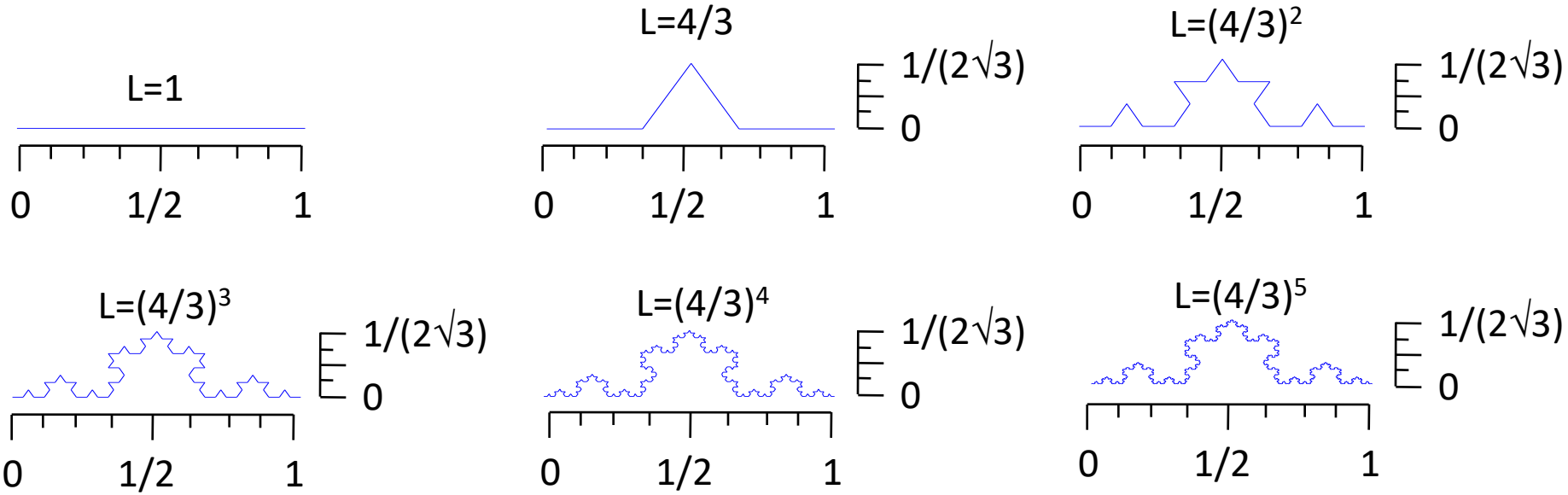


$$L = \sum_i \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_i \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

$$\Rightarrow \quad L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$$

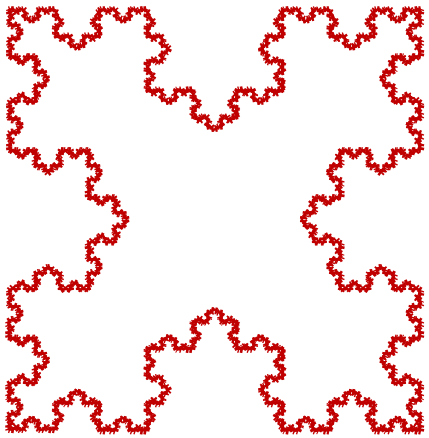
Basically, ignores details smaller than an increment ε and then considers what happen when $\varepsilon \to 0$.

For example, the Koch curve



The length of Koch curve is $\lim_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty$

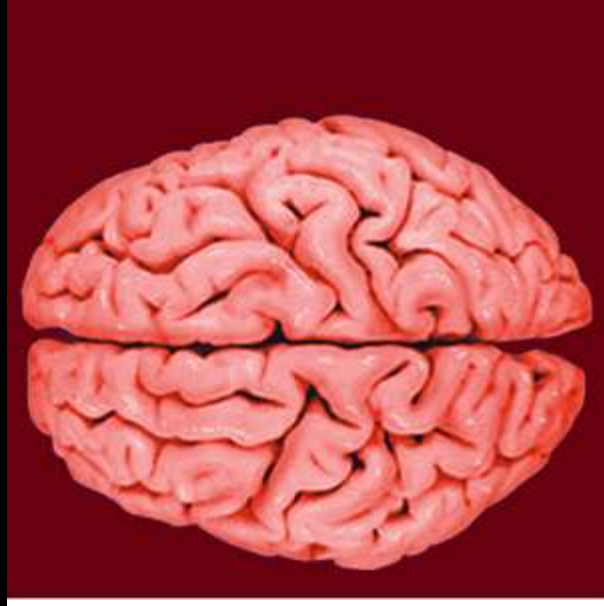
The area under Koch curve is $\sum_{n=1}^{\infty} N_T(n)A_T(n) = \frac{\sqrt{3}}{20}$



This figure has an infinite perimeter, but an area of only $\frac{5-\sqrt{3}}{5}$ m²

The length of the circulatory system is 80 millions meters!

The length of lung alveoli is 350000 meters!



The surface of human brain is 6 square meters!

Nature's frugality requires fractal structures

This was the big popular inspiration

Benoit Mandelbrot, *"How long is the coast of britain?"*, Science, Vol. 156, No. 3775. (May 5, 1967), pp. 636-638

"Why this text came to be written? It was intended to be a ''Trojan'' horse allowing a bit of mathematical esoterica to ''infiltrate'' surreptitiously hence near-painlessly, the investigation of the messiness of raw nature. Today, it means that everyone knows how to answer the question raised in this paper's title. And the notion of fractal dimension is very widely known and used."

How long is the coast of Britain?

Unit = 200 km,
Length = 2400 km

Unit = 100 km,
Length = 2800 km

Unit = 50 km,
Length = 3400 km

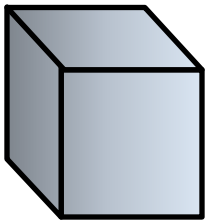
Dimensions

Informally, the dimension of an object is the minimum number of coordinates needed to specify each point within it

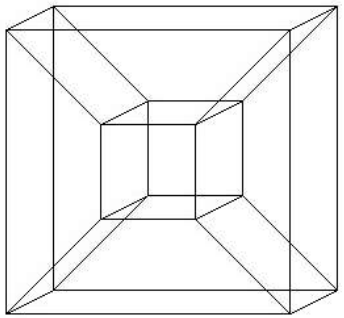
- Dimension 0 {0} — Measure : Number of points

- Dimension 1 (x) — Measure : Length

- Dimension 2 (x,y) — Measure : Area

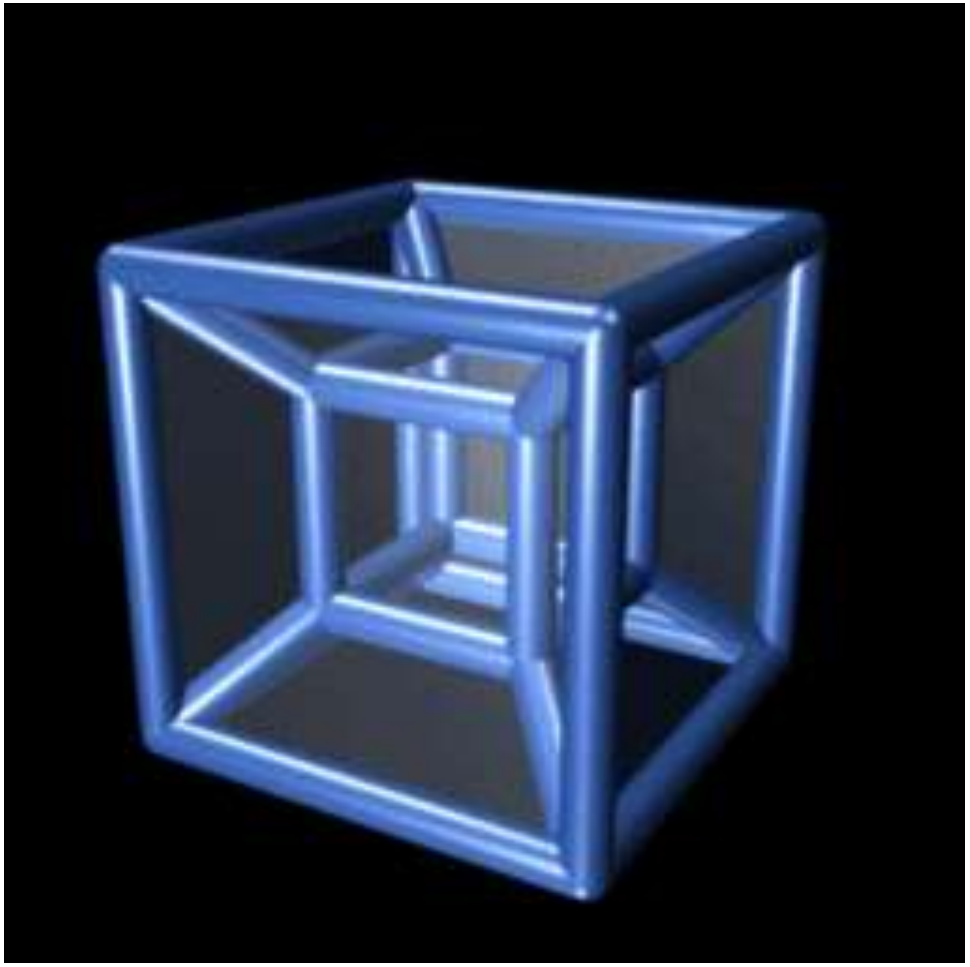


- Dimension 3 (x,y,z) — Measure : Volume

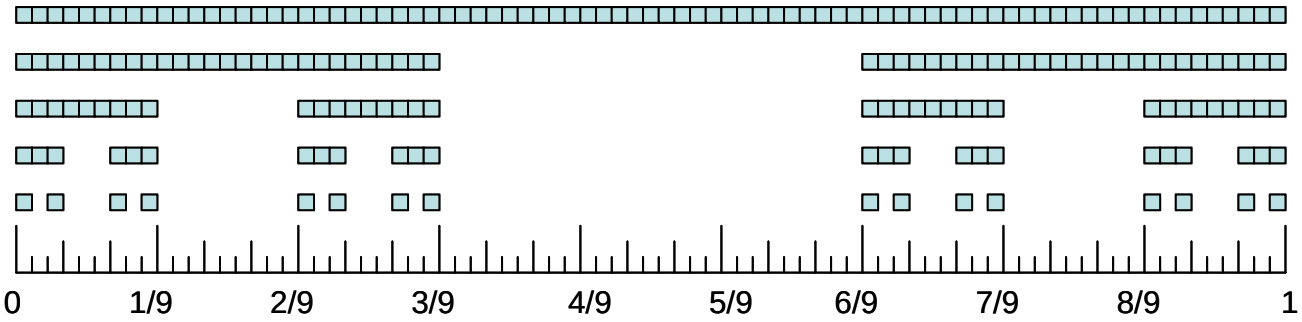


- Dimension 4 (w,x,y,z) Tesseract
  (Shadow of a 4-dimensional hypercube)

Tesseract

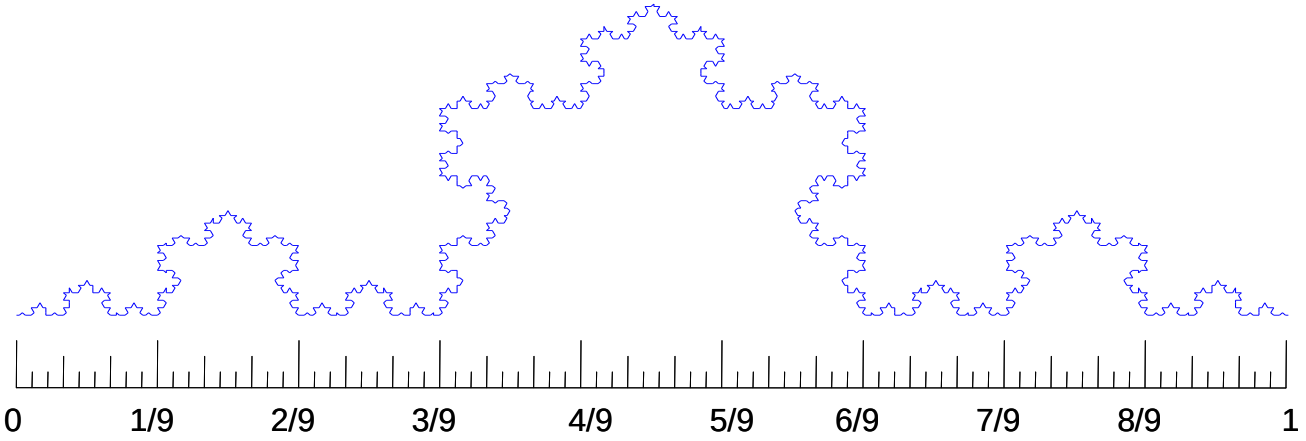


Self-similar fractals



The Cantor set is composed of ISOLATED POINTS in the interval [0, 1], but it has as many elements as the whole CONTINUOUS interval!

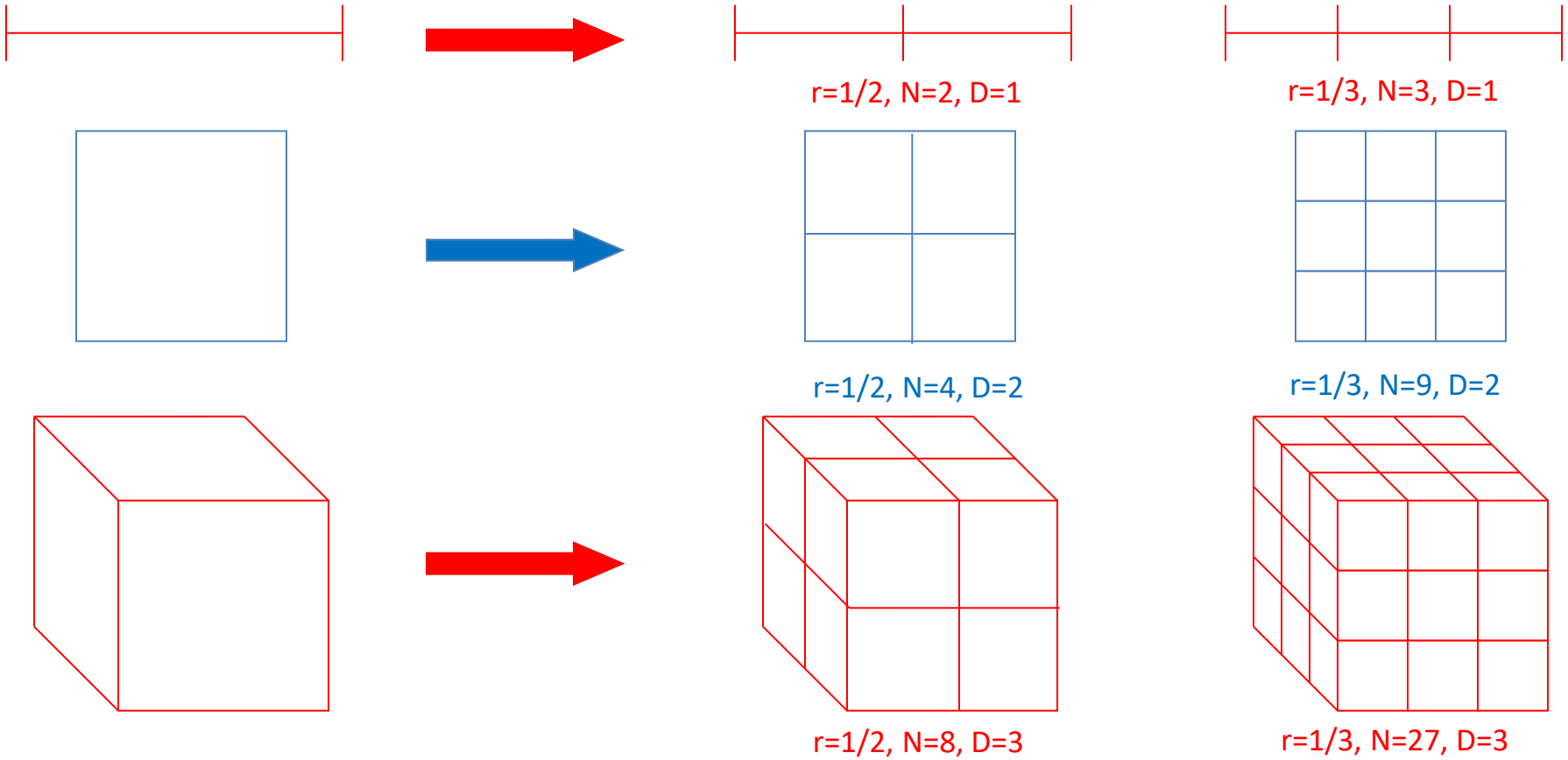
The dimension of the Cantor set cannot be 1, but should be greater than zero, since its length is zero and its number of points is uncountable infinity



The area under the Von Koch curve is only $(\sqrt{3})/20$ m^2, but it has an infinite length

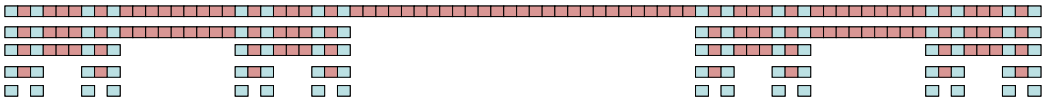
The dimension of the Von Koch curve cannot be 2, but should be greater than one, since its area is zero and its length is infinity

(One) Concept of Dimension (through self-similarity)

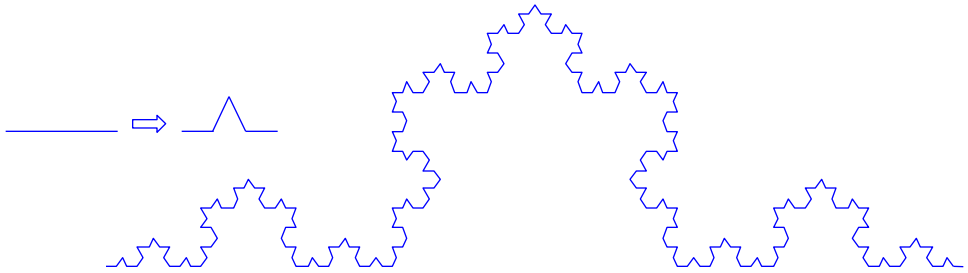


$$N = r^{-D} \Rightarrow D = \log(N) / \log(1/r)$$

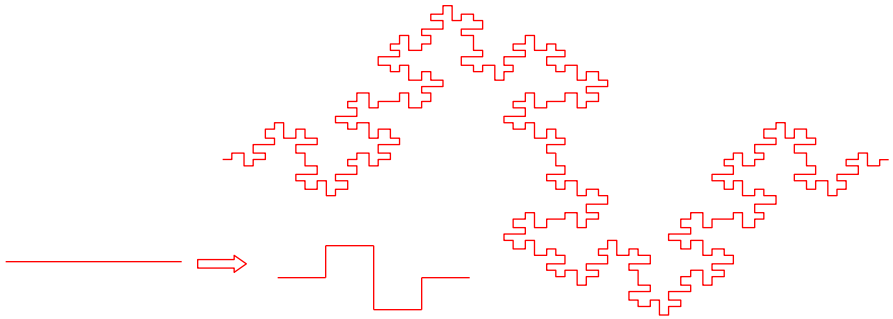
Dimension of self-similar fractal figures



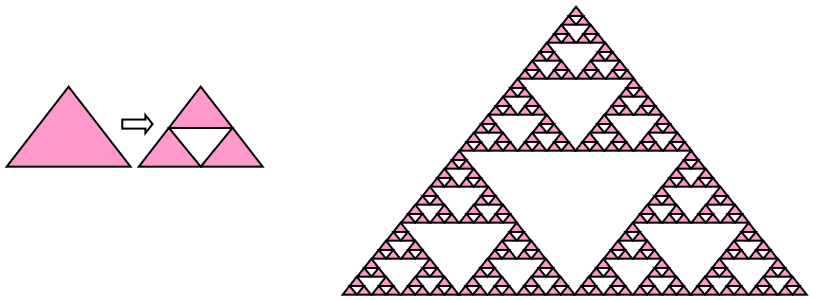
$N = 2,$
$r = 1/3,$
$D = log(2)/log(3) = 0.63$



$N = 4,$
$r = 1/3,$
$D = log(4)/log(3) = 1.26$



$N = 8,$
$r = 1/4,$
$D = log(8)/log(4) = 1.5$

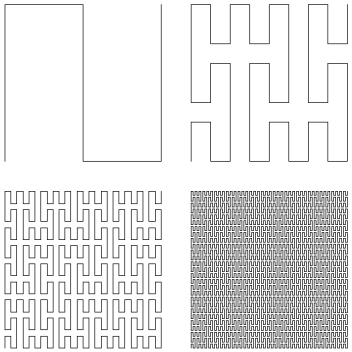


$N = 3,$
$r = 1/2,$
$D = log(3)/log(2) = 1.58$

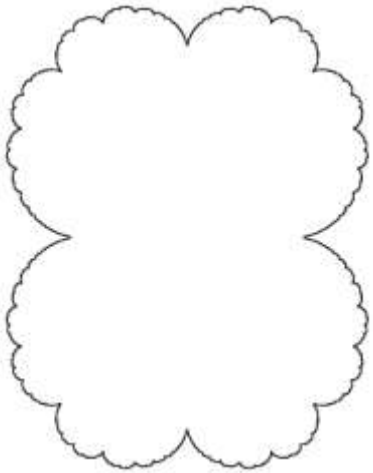
Dimension of some fractals

Cantor set : 0.631

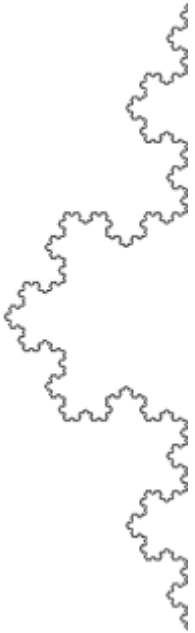
Cantor set in 3D : 1.893



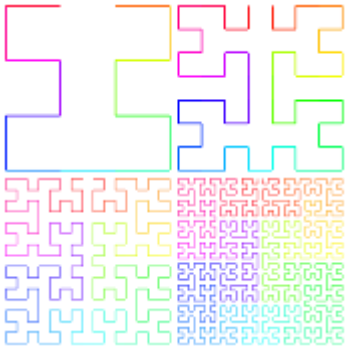
Peano curve : 2



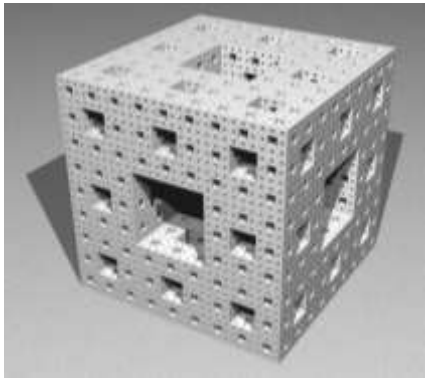
Julia set(1/4) : 1.081



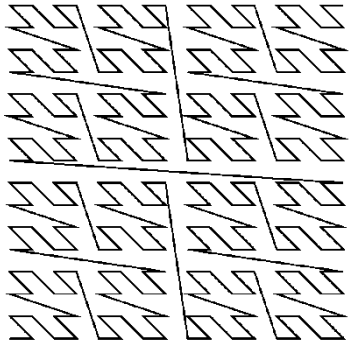
Von Koch curve : 1.262



Moore curve : 2



Menger sponge : 2.727



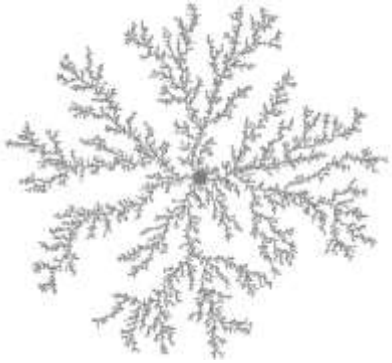
Z-order curve : 2

Lorenz atractor : 2.06

Dimension of some natural fractals

coastline of Great Britain : 1.25

Coastline of Norway : 1.52

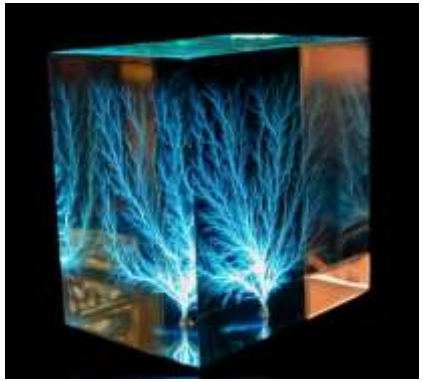


Diffusion limited aggregation : 1.7

Galaxy clusters : 2

Cauliflower : 2.33

Crumpled paper : 2.5



Lichtenberg lamp : 2.5

Broccoli : 2.66

Surface of human brain : 2.79

Lung alveoli : 2.97

How can these dimensions be measured or computed if the objects are not exactly self-similar?

The concept: Hausdorff dimension

- Previous definitions of
  - Euclidean dimension
  - self-similar dimension
- … are simply versions of the true “fractal dimension”
  - **Hausdorff dimension**
- To begin with, let us define the Hausdorff measure

δ-cover of a set

- If U is any non-empty subset of the n-dimensional Euclidian space, $\mathbb{R}^n$, …
- … the diameter of U is defined as

$$|U| = \sup\{|x - y| : x, y \in U\}$$

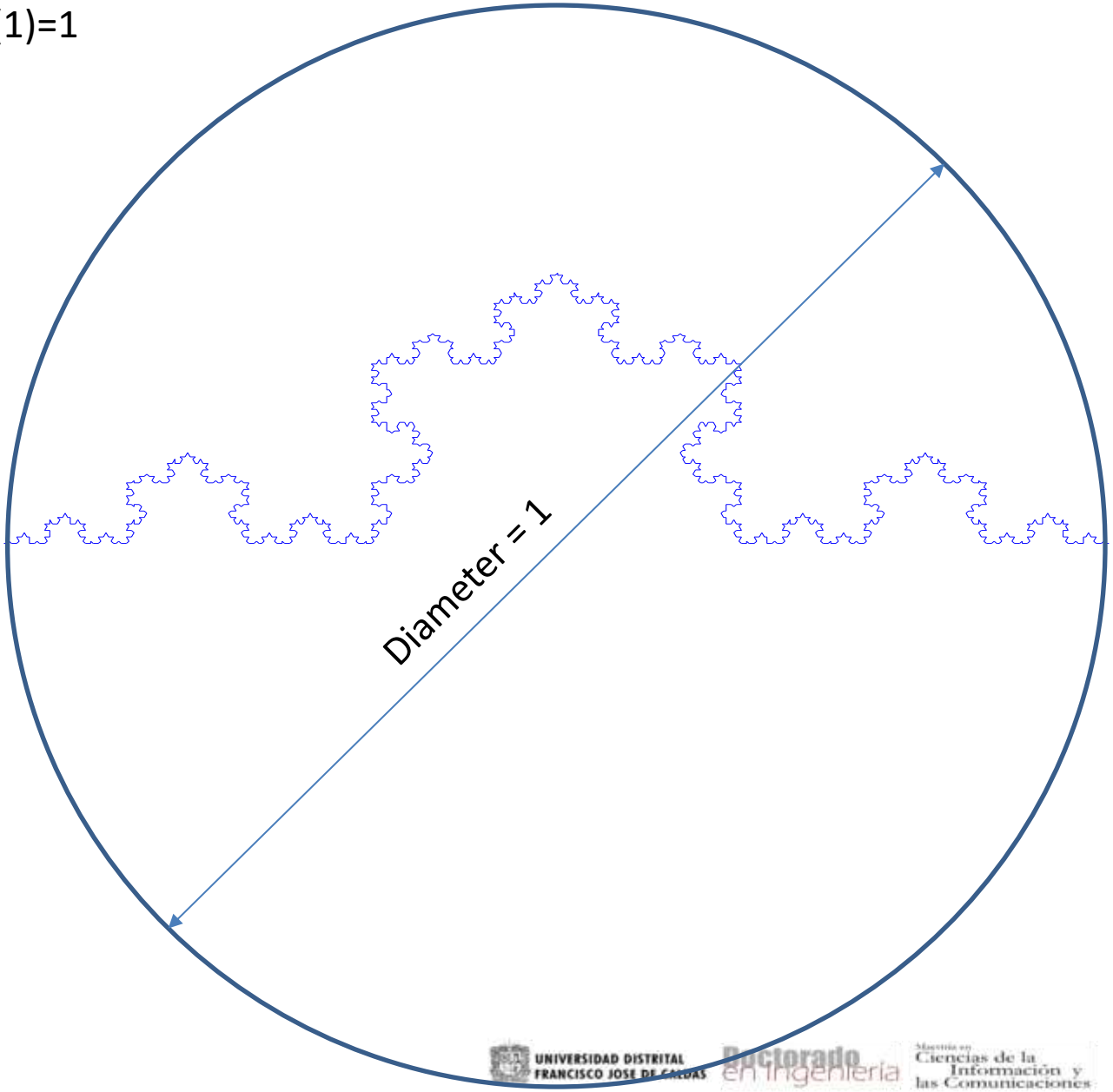
- If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e.,

$$F \subset \bigcup_{i=1}^{\infty} U_i$$

- … with $0 \leq |U_i| \leq \delta$ for each i, …
- … we say that $\{U_i\}$ is a δ-cover of F

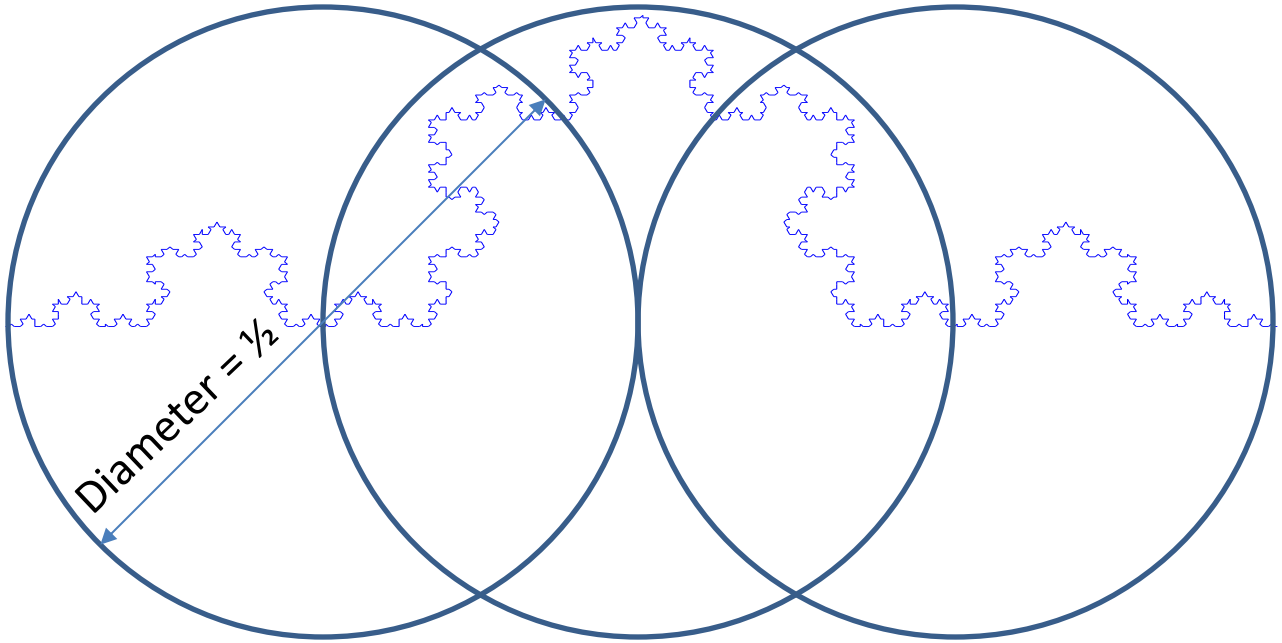
1-cover of the Koch curve

N(1)=1



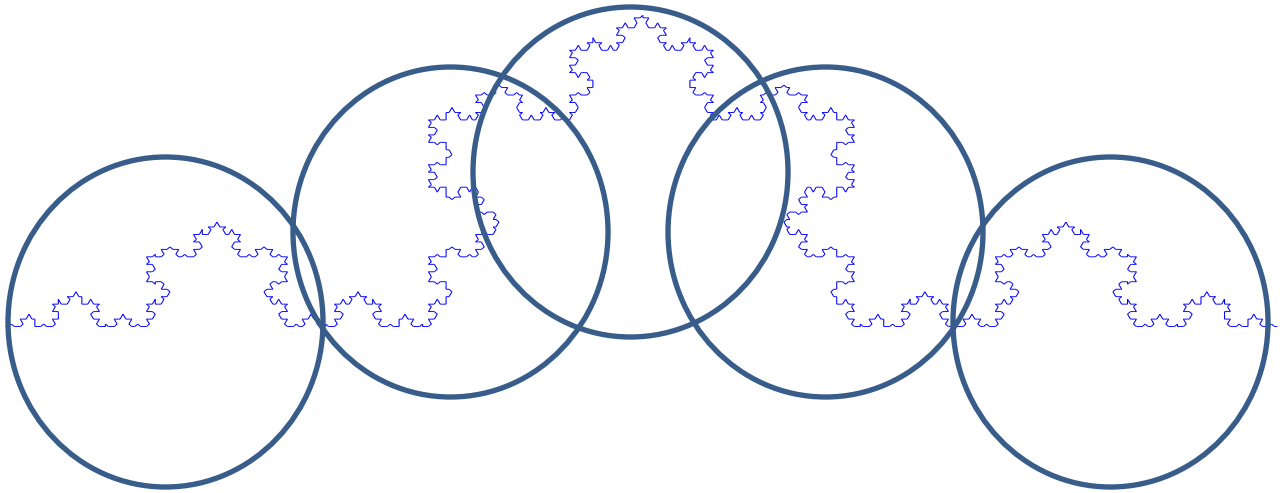
½ - cover of the Koch curve

N(½)=3



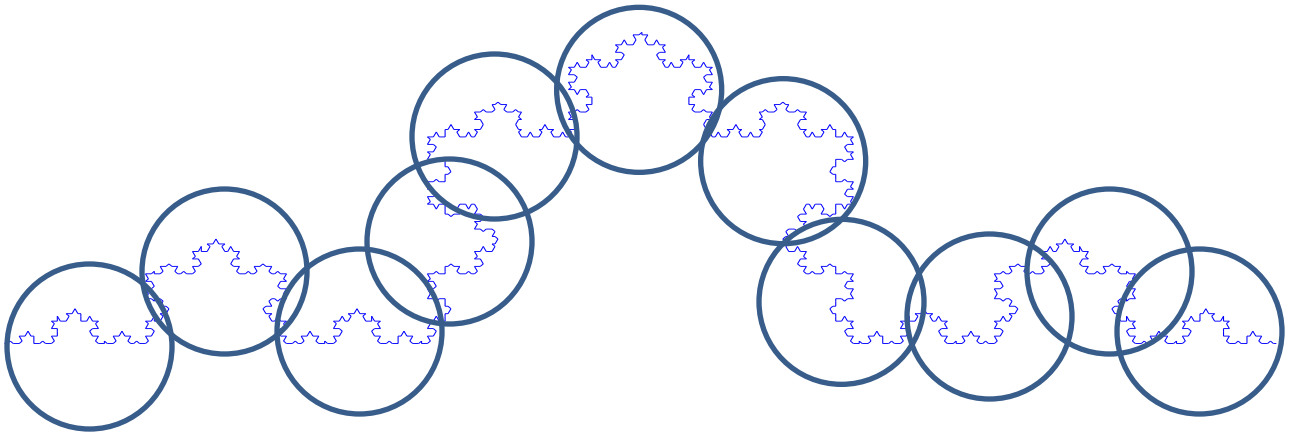
¼ - cover of the Koch curve

N(¼)=5



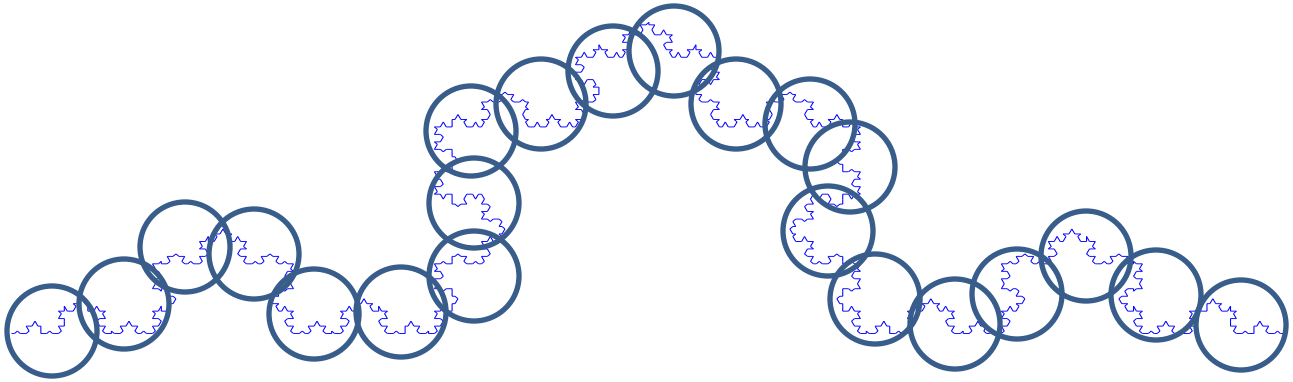
1/7 - cover of the Koch curve

N(1/7)=11



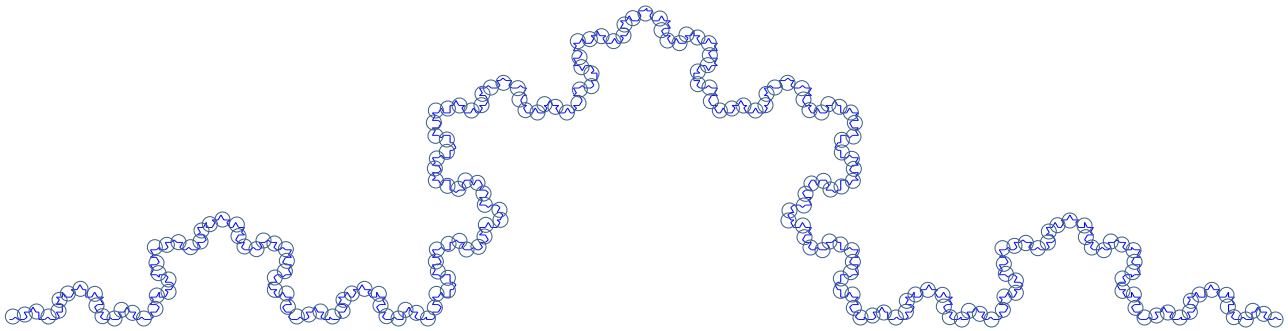
1/12 - cover of the Koch curve

N(1/12)=22



δ - cover of the Koch curve, δ = 1/88

N(1/88)=244



Hausdorff measure

- Suppose F is a subset of $\mathbb{R}^n$, and s is a non-negative number. For any $\delta>0$ we define

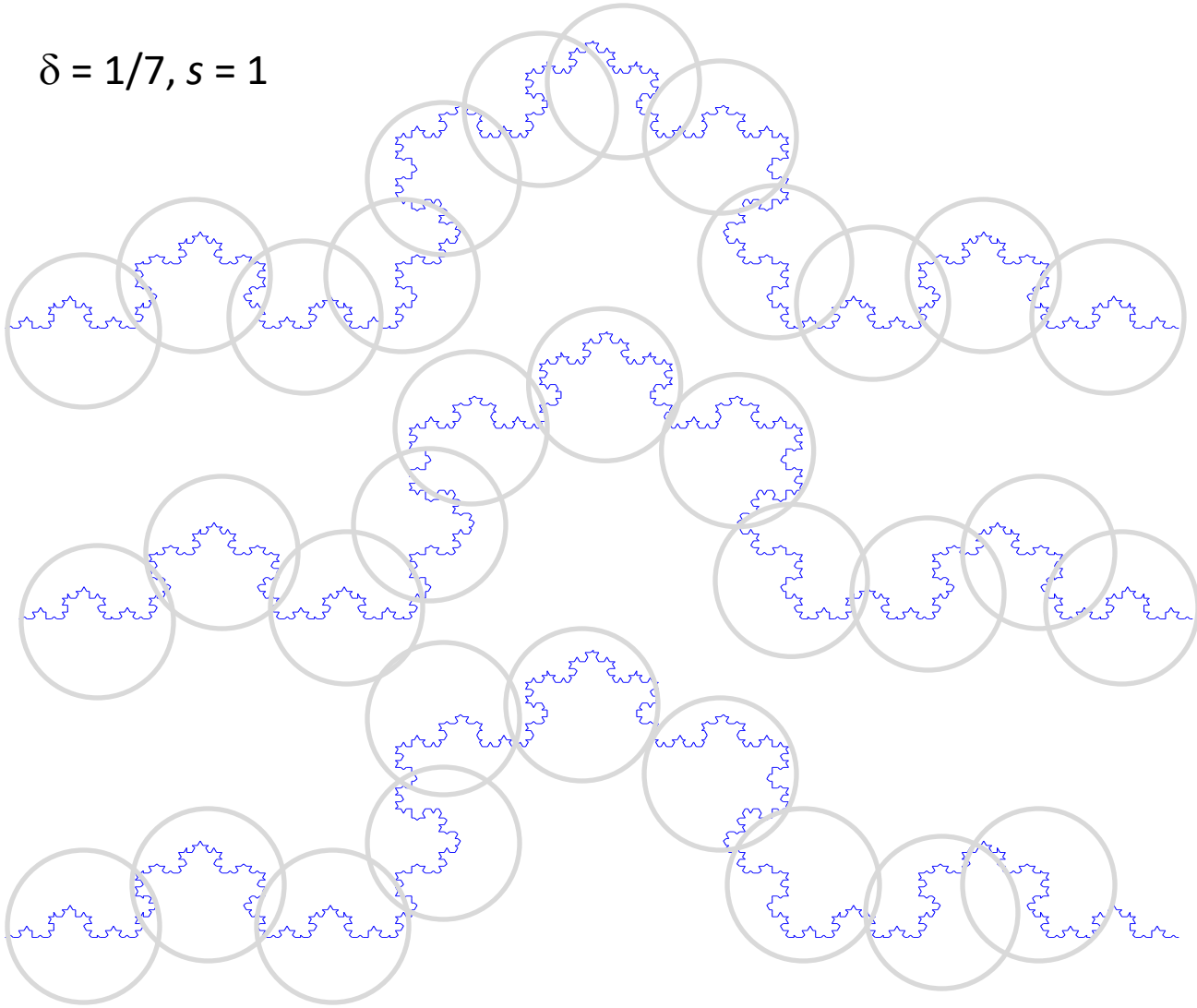
$$H^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

- As δ decreases, the class of permissible covers of F is reduced, so the infimum increases and, so, approaches a limit as $\delta\to 0$:

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$$

s-dimensional Hausdorff measure of set *F*

$\delta = 1/7$, $s = 1$



$$\sum_{i=1}^{12} \delta^1 = 12\delta \quad \checkmark$$

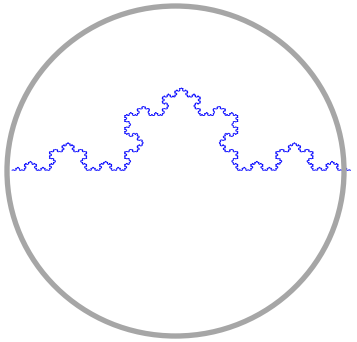
$$\sum_{i=1}^{11} \delta^1 = 11\delta \quad \checkmark$$

$$\sum_{i=1}^{10} \delta^1 = 10\delta \quad \times$$

$$H^1_{1/7}(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^1 : \{U_i\} \text{ is a } (1/7)\text{-cover of } F\right\} = \frac{11}{7}$$

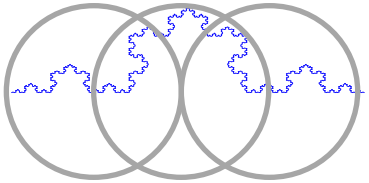
$$H_1^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (1)-cover of }F\right\}=1\cdot 1^1=1$$

$$H_1^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (1)-cover of }F\right\}=1\cdot 1^2=1$$



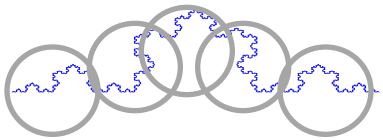
$$H_{0.5}^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (0.5)-cover of }F\right\}=3\cdot 0.5^1=1.5$$

$$H_{0.5}^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (0.5)-cover of }F\right\}=3\cdot 0.5^2=0.75$$



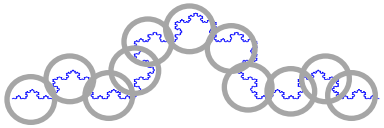
$$H_{0.25}^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (0.25)-cover of }F\right\}=5\cdot 0.25^1=1.25$$

$$H_{0.25}^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (0.25)-cover of }F\right\}=5\cdot 0.25^2=0.3125$$



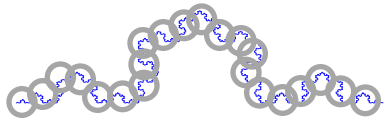
$$H_{0.143}^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (0.143)-cover of }F\right\}=11\cdot 0.143^1=1.573$$

$$H_{0.143}^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (0.143)-cover of }F\right\}=11\cdot 0.143^2=0.2245$$



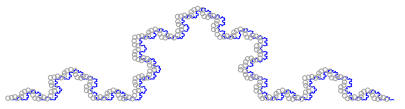
$$H_{0.083}^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (0.083)-cover of }F\right\}=22\cdot 0.083^1=1.83$$

$$H_{0.083}^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (0.083)-cover of }F\right\}=22\cdot 0.083^2=0.153$$

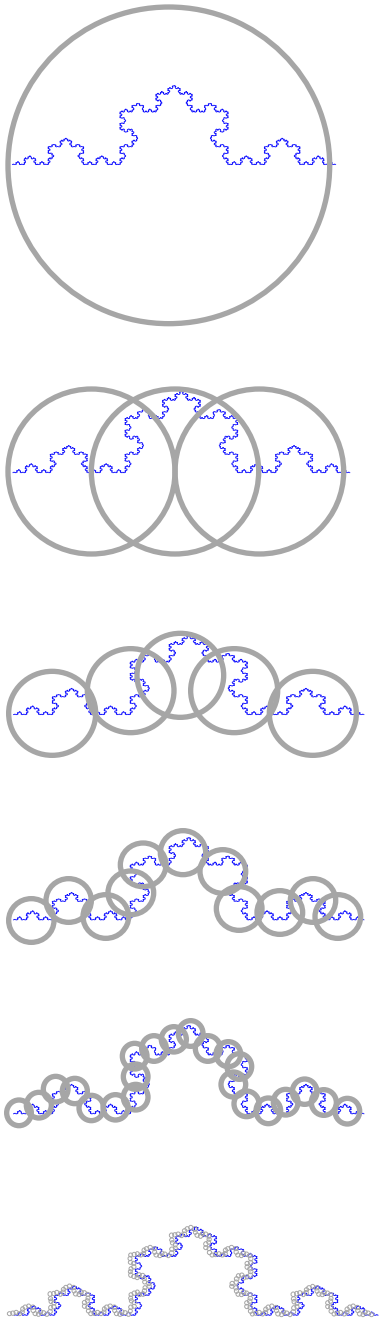


$$H_{0.0114}^1(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^1:\{U_i\}\text{ is a (0.0114)-cover of }F\right\}=244\cdot 0.0114^1=2.773$$

$$H_{0.0114}^2(F)=\inf\left\{\sum_{i=1}^{\infty}|U_i|^2:\{U_i\}\text{ is a (0.0114)-cover of }F\right\}=244\cdot 0.0114^2=0.0315$$



δ	H_δ^1(Von Koch)	H_δ^2(Von Koch)
1	1	1
1/2	3/2	3/4
1/4	5/4	5/16
1/7	11/7	11/49
1/12	22/12	22/144
1/88	244/88	244/7744
0	**∞**	**0**



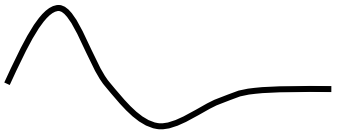
The Hausdorff measure

$$H^s(F) = \lim_{\delta \to 0} H_\delta^s(F)$$

Can (and use to) be 0 or ∞

$H^0(F) = \text{Number of points}$

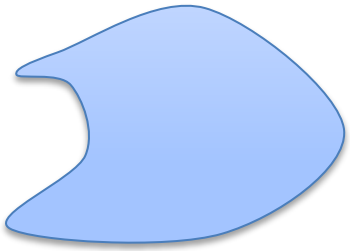
$H^1(F) = 0$



$H^0(F) = \infty$

$H^1(F) = \text{Length of the curve}$

$H^2(F) = 0$



$H^1(F) = \infty$

$H^2(F) \propto \text{Area of the surface}$

$H^3(F) = 0$

$H^2(F) = \infty$

$H^3(F) \propto \text{Volume of the body}$

$H^4(F) = 0$

Towards Hausdorff dimension

- Since

$$H_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

- for any $\delta<1$, $H_d^s(F)$ does not increase with s. Consequently

$$H^s(F) = \lim_{\delta\to 0} H_\delta^s(F)$$

does not increase with s.

- Furthermore, since

$$\sum_i |U_i|^t = \sum_i |U_i|^{t-s}|U_i|^s \le \delta^{t-s}\sum_i |U_i|^s$$

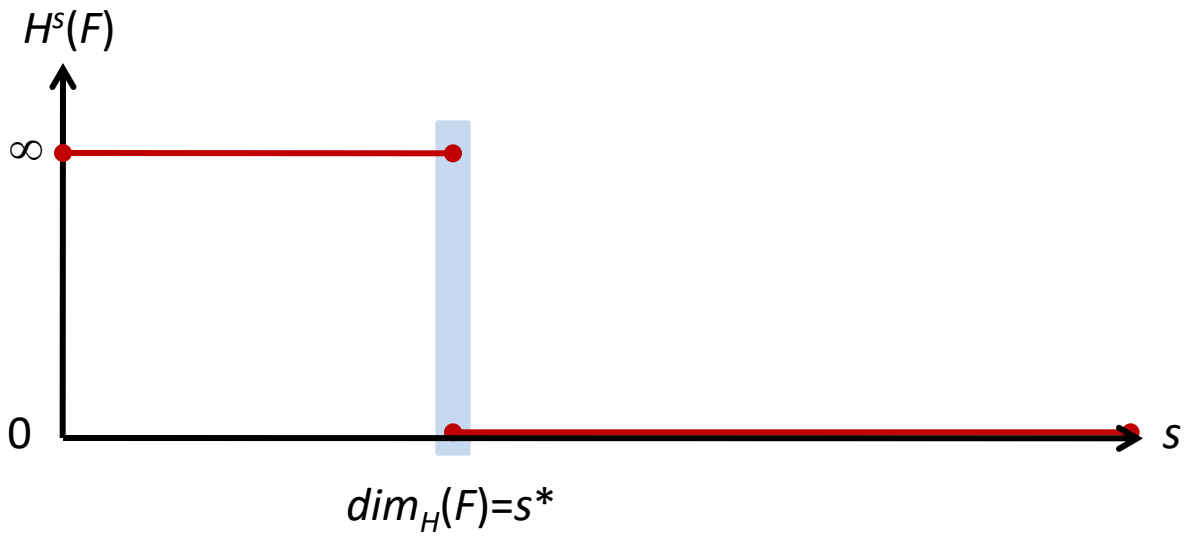
- the infimum must also obey this relation

$$H_\delta^t(F) \le \delta^{t-s} H_\delta^s(F)$$

- So, in the limit,

if $H^s(F)<\infty$, $H^t(F)=0\ \forall\ t>s$

Hausdorff dimension

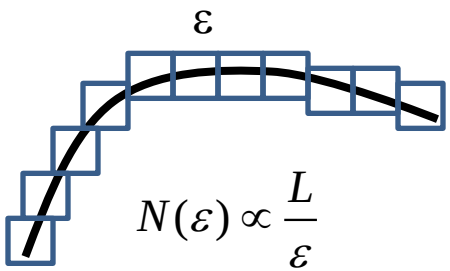


$$\dim_H(F) = \inf\left\{s \geq 0 : H^s(F) = 0\right\} = \sup\left\{s \geq 0 : H^s(F) = \infty\right\}$$

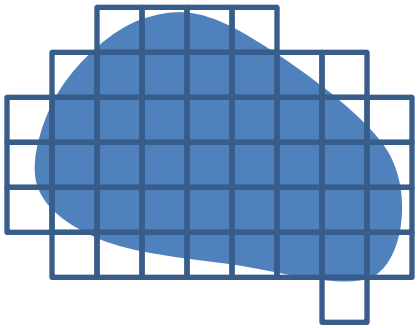
Does not say anything about $H^{s^*}(F)$. It could be 0, ∞, or any intermediate number, but it is normally too hard to compute. Many (MANY) recent math papers still address the Hausdorff measure of the Cantor set, which came up to be $H^{s^*}(\mathcal{C}) = 1$.

Box dimension

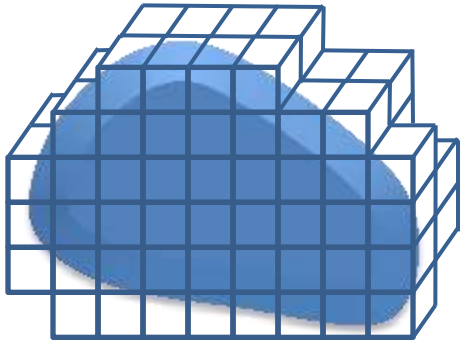
- Cover the set with boxes of size ε



$$N(\varepsilon) \propto \frac{L}{\varepsilon}$$



$$N(\varepsilon) \propto \frac{A}{\varepsilon^2}$$



$$N(\varepsilon) \propto \frac{V}{\varepsilon^3}$$

- As with smooth lines, surfaces and solid objects, one can expect the dimension of a set to equals the exponent d in the power law

$$N(\varepsilon) \propto \frac{1}{\varepsilon^d}$$

Box dimension

$$d = \lim_{\varepsilon \to 0} \frac{\ln\left(N(\varepsilon)\right)}{\ln\left(1/\varepsilon\right)}$$

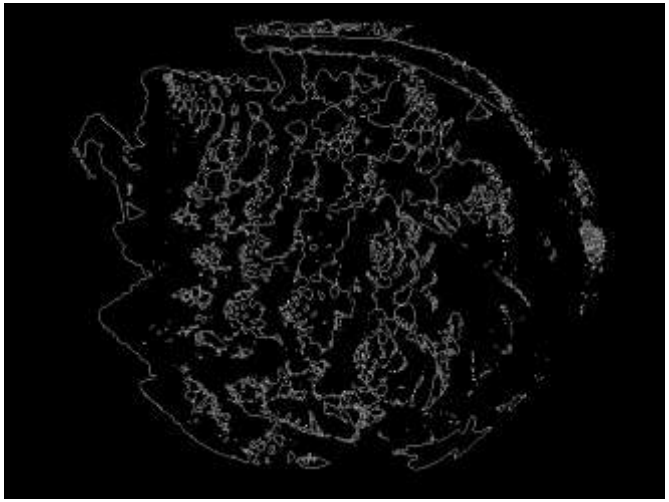
- Find the box dimension of the Cantor set

  - Let S_n be the set at the n^{th} step of Cantor set construction.
  - For each n, the 2^n intervals of size 3^{-n} of S_n cover the Cantor set.
  - Using $\varepsilon = 3^{-n}$, we need $N(\varepsilon)=2^n$.
  - As $n \to \infty$, $\varepsilon \to 0$
  - $d = \lim_{n\to\infty} \frac{\log(2^n)}{\log(3^n)} = \frac{n\log(2)}{n\log(3)} = \frac{\log(2)}{\log(3)} = 0.631$

Finding the box-dimension of a 2D fractal set

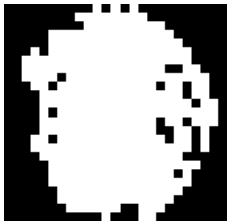
```
% Carga y despliega la imagen original
p = imread('Fractal_Broccoli.jpg');
figure(1); imshow(p)
```

```
% detecta los bordes sobre la imagen B&N
p  = edge(double(im2bw(p,graythresh(p))));
figure(2); imshow(p)
```



```
[Nx, Ny] = size(p);  % Tamaño de la imagen
nBloques = [25 50 75 100 125 150]'; % Número de bloques
NB = length(nBloques);
tabla = zeros(NB,1);
for fg = 1:NB
    tamanoBloque_x = floor(Nx./nBloques(fg));
    tamanoBloque_y = floor(Ny./nBloques(fg));
    ocupado = zeros(nBloques(fg),nBloques(fg));
    for i = 1:nBloques(fg)
        xi = (i-1)*tamanoBloque_x + 1;
        xf   = i*tamanoBloque_x;
        for j = 1:nBloques(fg)
            yi = (j-1)*tamanoBloque_y + 1;
            yf   = j*tamanoBloque_y;
            bloque = p(xi:xf,yi:yf);
            ocupado(i,j) = any(bloque(:));
        end
    end
    tabla(fg) = nnz(ocupado);
    figure(3); subplot(2,3,fg); imshow(ocupado);
end
```

eps = 0.04, N(eps) = 369



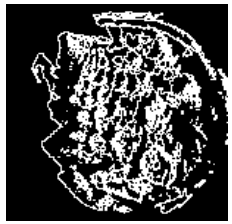
eps = 0.02, N(eps) = 1114

eps = 0.0133, N(eps) = 2055

eps = 0.01, N(eps) = 3091

eps = 0.008, N(eps) = 4082

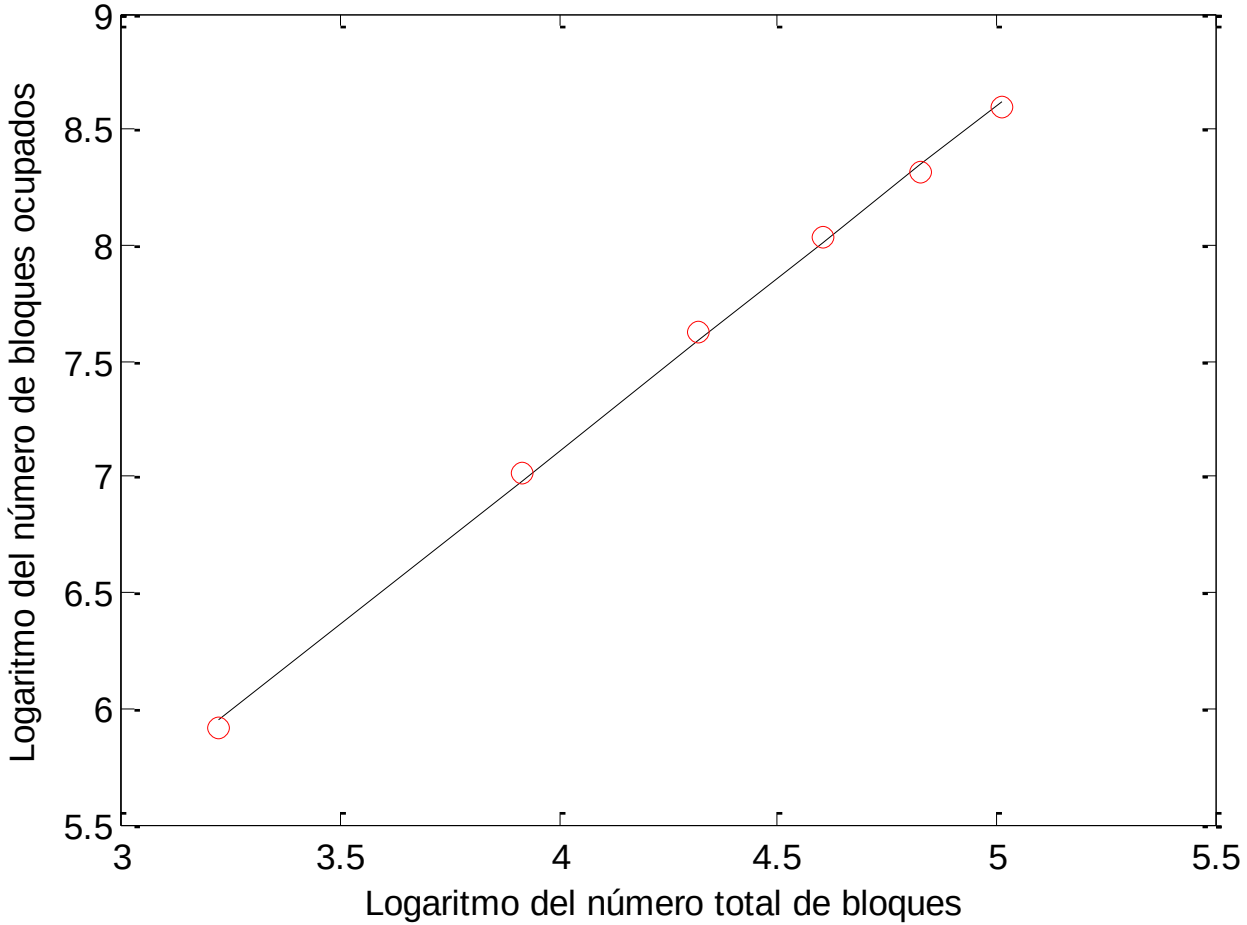
eps = 0.0067, N(eps) = 5430



Finding the box-dimension of a 2D fractal set

```
x = log(nBloques);
y = log(tabla);
p = polyfit(x,y,1);
y1 = polyval(p,x);
figure(4); plot(x,y1,'k-',x,y,'ro')
Dimension_de_Hausdorff = p(1)
```

Dimension_de_Hausdorff = 1.4921



HOMEWORK # 4

1. The given files include 12 textures (from `textura01.bmp` to `textura12.bmp`) and 12 natural objects (3 flowers (or flower seeds), 3 clouds, 3 wood textures and 3 iris patterns, from `prueba01.bmp` to `prueba12.bmp`), as shown in Fig. 2. Compute the box-counting dimension of each image and discuss whether this parameter is a good discriminator to tell woods from clouds from flowers from irises. Could you classify some of the original textures within your four classes?

   Note 1. So far, we have used the box-counting dimension algorithm to compute the dimension of a rough line in a plane. But in this homework we have to compute the dimension of a rough surface in a 3D volume (the grey level gives the third Euclidean dimension), so we have to count 3D boxes, not 2D squares*.

   Note 2. If you have a set of geometrical 3D surfaces of your own in which you are interested because of your research project, please feel free to use it. All you have to do is to compute the box-counting dimension and decide if it is a good discriminator for different classes within your set (but using a surface in a 3D space instead of a line in a 2D space).

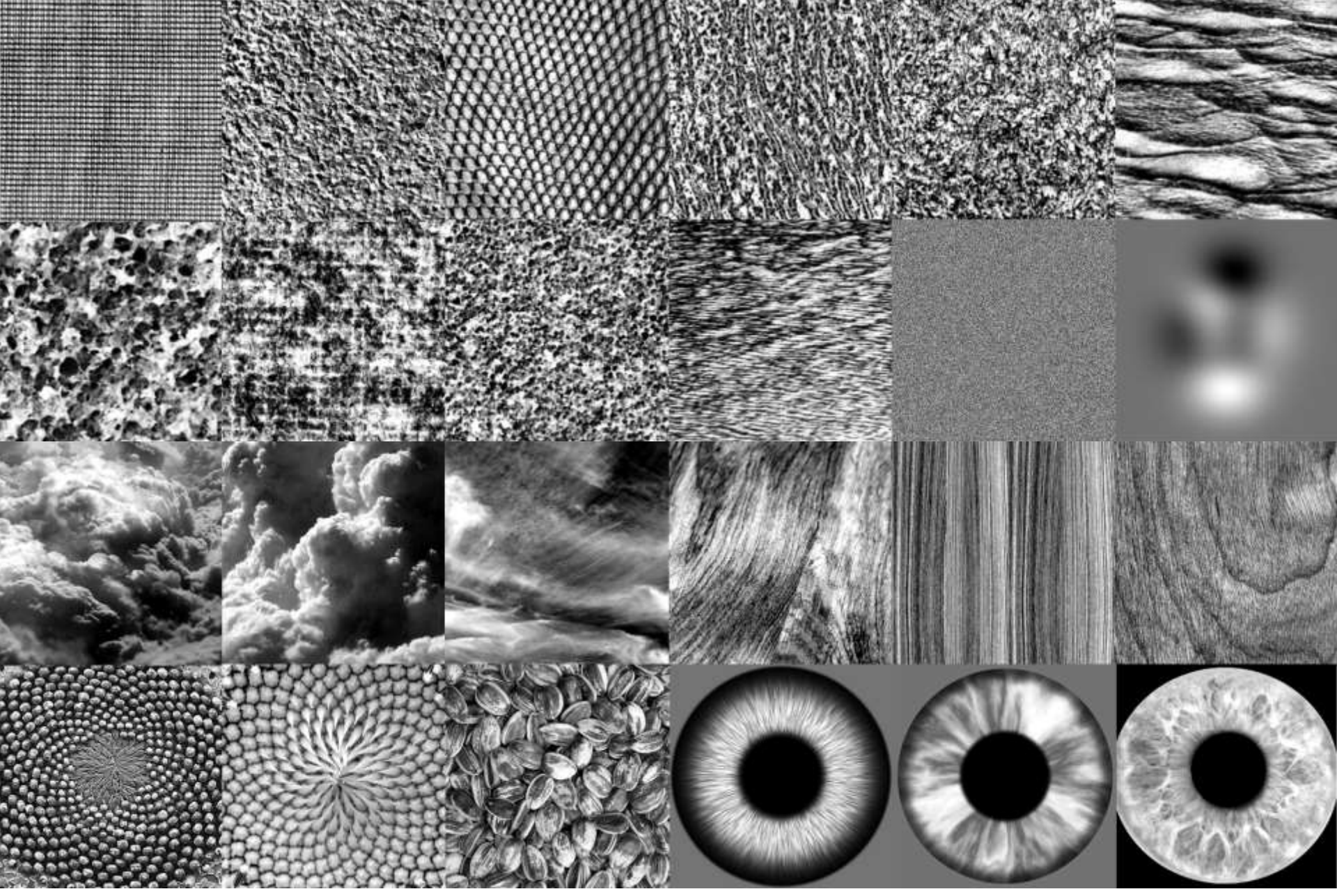
2. Compute the box-counting dimension of (a) a finite set of isolated points in the unit interval, (b) the unit interval in the real line and (c) the unit square in the plane.

3. What is the self-similar dimension of the set that results from taking out from the closed unit interval all the open sub-intervals whose decimal expansion excludes the digit 5? (in the expansion, a 5 followed by an infinite number of 0's is changed by a 4 followed by an infinite number of 9's)

4. Starting with the unit interval, $I_0=[0,1]$, we define the following IFS

   $$I_{n+1} = \{x/2: x \in I_n\} \cup \{(3+2x)/5: x \in I_n\}$$

   What is the Hausdorff dimension of the set of points that is left after an infinite number of iterations?



---

* Last class, we used the BoxCounting algorithm to compute the dimension of a rough line in a plane. But if we consider a grey level 2D image as a 2D surface in a Euclidean 3D space, where the grey level gives the third Euclidean dimension, we could use the same method to count the boxes. The following text is taken from [Jian Lia, Qian Dub and Caixin Suna. "An improved box-counting method for image fractal dimension estimation" Pattern Recognition 42(2009) 2460-2469]:

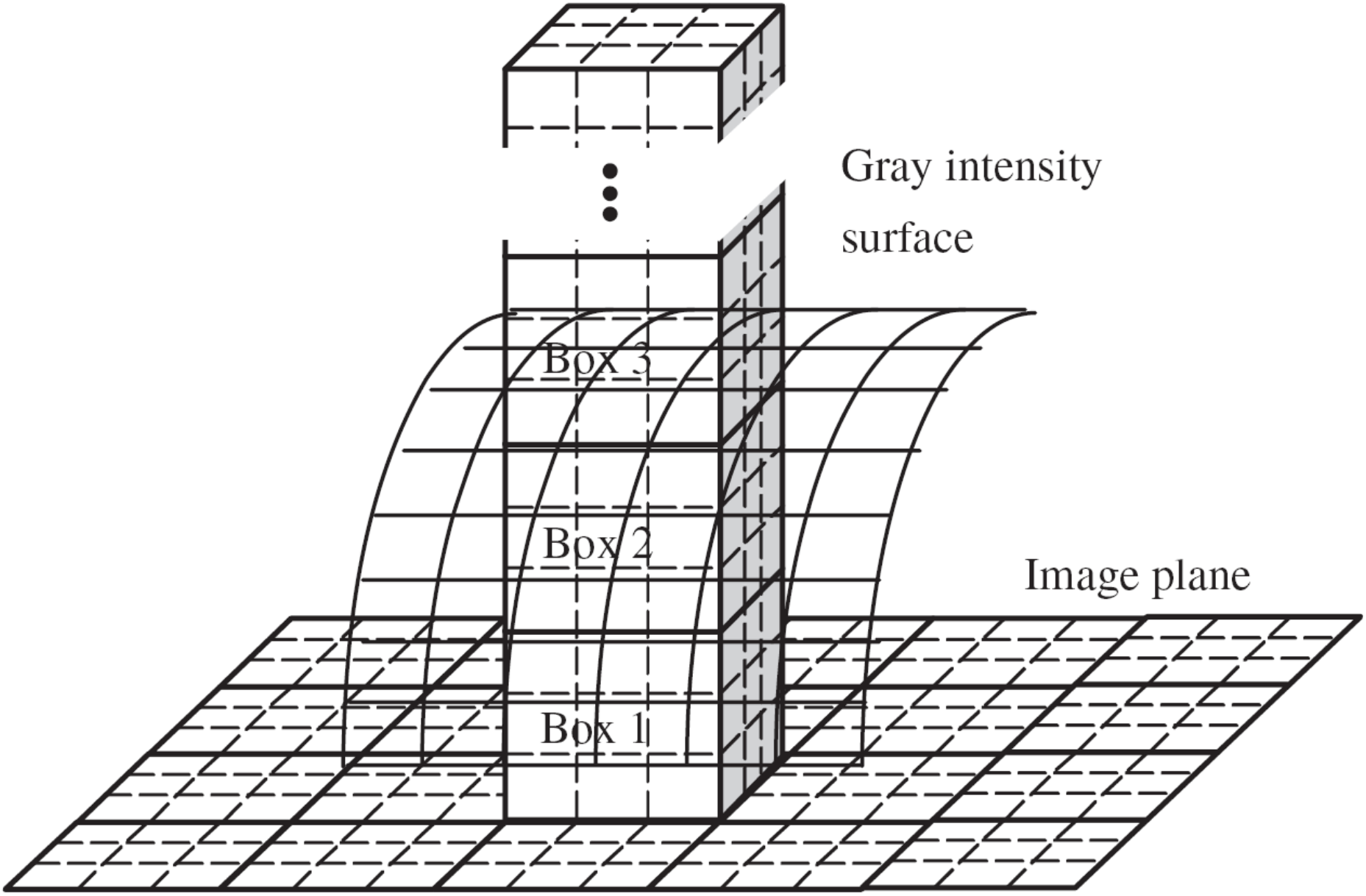
"Consider an image of size $M \times M$ as a 3D spatial surface with (x,y) denoting pixel position on the image plane, and the third coordinate (z) denoting pixel gray level. In the DBC method, the (x,y) plane is partitioned into non-overlapping blocks of size $s \times s$, where $1 < s \leq M/2$ and s is an integer. N_s is counted in the DBC method using the following procedure. On each block there is a column of boxes of size $s \times s \times s'$, where s' is the height of each box, $G/s' = M/s$, and G is the total number of gray levels. For example, $s = s' = 3$ in Fig. 1. Assign numbers 1,2,...,to the boxes as shown in Fig. 1. Let the minimum and maximum gray level in the $(i, j)^{th}$ block fall into the k^{th} and l^{th} boxes, respectively. The boxes covering this block are counted in the number as

$$n_s(i, j) = l - k + 1$$

where the subscript s denotes the result using the scale s. For example, $n_s(i, j) = 3 - 1 + 1$ as illustrated in Fig. 1. Considering contributions from all blocks, N_s is counted for different values of s as

$$N_s = \sum_{i,j} n_s(i, j)$$

Then the BCD can be estimated from the least squares linear fit of $\log(N_s)$ versus $\log(1/s)$".


Gray intensity
surface
Box 3
Box 2
Image plane
Box 1