Homework #5

A. Simple Random Process

Let $\{X_i,\ i=1,2,\ldots\}$ be an iid (independent and identically distributed) Bernoulli process. This is, each X_i is a Bernoulli random variable with $P[X_i=1]=p$ and $P[X_i=0]=1-p$, independent of the other random variables in the process, for some real number $0 < p < 1$.

Let $\{Y_n,\ n=1,2,\ldots\}$ be a random walk constructed according to the previous iid Bernoulli process, as follows:

$$Y_n = \sum_{i=1}^{n} 2X_i - 1, \quad n = 1, 2, \ldots$$

This is, at each discrete instant of time, n, a particle takes a step to the left if $X_n = 0$ or to the right if $X_n = 1$. In other words, starting at $Y_0=0$, the random walk goes as $Y_n = Y_{n-1} + (2X_n-1)$.

1. Find the first and second order distribution of the random walk $\{Y_n,\ n=1,2,\ldots\}$.
2. Show that the first and second order distributions are enough for completely describing the random walk $\{Y_n,\ n=1,2,\ldots\}$.
3. Find the mean value function, the autocorrelation function and the autocovariance function

$$\mu_n = E\left[Y_n\right], \quad R_{m,n} = E\left[Y_m Y_n\right], \quad C_{m,n} = E\left[\left(Y_m - \mu_m\right)\left(Y_n - \mu_n\right)\right]$$

B. Fractional Brownian Motion

In class we defined the FBM random process (fractional brownian motion) as a stationary increment process $\{Y(t),\ t\in\mathbb{R}\}$ for which $Y(t)-Y(s)\approx\mathcal{N}(0,\sigma^2|t-s|^{2H})\ \ \forall\ t>s$, with $E[Y(t)]=0\ \forall\ t\in\mathbb{R}$ and $Y(0)=0$. We noticed it is a self-similar process in the sense that

$$Y(t) \overset{d}{=} a^{-H} Y(at)$$

1. Show that $R_Y(s,t) = E[Y(s)Y(t)] = \sigma^2(t^{2H} + s^{2H} - |t-s|^{2H})/2$
2. Let $\{X(t) = Y(t) - Y(t-1),\ t\in\mathbb{R}\}$ be the increment process of the FBM $\{Y(t),\ t\in\mathbb{R}\}$. Show that $R_x(t) = E[X(s)X(s+t)] = \sigma^2(|t+1|^{2H} + |t-1|^{2H} - 2|t|^{2H})/2$

C. Natural Fractal Processes

Consider the two signals TiempoEntrePaquetes and TiempoEntreDisparos. The first one is a list of the interarrival times between consecutive packets that arrive to a local area network, and the second one is a list of the time between neuron firings that occur in an auditive nerve.

1. Using the variance-time plot, estimate the Hurst parameter of each trace using the scales in the range m=2.^(0:11) for the neuron firing trace and in the range m=2.^(0:14) for the packet arrival trace. Repeat the estimation above using the ranges of scales m_0=2.^(0:3), m_1=2.^(3:5) and m_2=2.^(5:11) for the neuron firing trace, and the ranges m_0=2.^(0:6) and m_1=2.^(6:14) for the packet arrival trace. Discuss the results.
2. Divide the packet signal in blocks of 128 samples. Randomly permute the samples within each block, keeping the original order of the blocks. Using the variance-time plot, estimate the Hurst parameter using scales in the range m_0=2.^(0:4) and m_1=2.^(8:14). Now keep the original order of the samples within each block and randomly permute the blocks. Using the variance-time plot, estimate the Hurst parameter using scales in the range m_0=2.^(0:7) and m_1=2.^(7:14). Discuss the results.

D. Power Laws and graphs

We have talked about power laws as sources/signatures of complex dynamics (Haussdorf dimension, Hurst parameter, 1/f noise, LRD, heavy tailed distributions, etc.). One of those power laws occurs regarding the topology of natural networks: Most nodes have few connections and few nodes have most connections. These network structures have important consequences on the dynamics of the associated system: Transportation networks, communication networks, metabolic networks, epidemiologic networks, social networks, etc. Since most of your research topics could benefit from the correct modeling of the underlying network, we are going to review some related concepts.

1. Reading assignment

   Light version: Albert-László Barabási and Eric Bonabeau "Scale-free Networks", Scientific American, May 2003

   Heavy version (if you get really interested): Réka Albert and Albert-László Barabási "Statistical mechanics of complex networks", Reviews of Modern Physics, Volume 74, January 2002

2. Synthetic networks

   Build two networks. Begin with two connected nodes and, at each time, add another node and connect it randomly to one of the existing nodes. For the first network, choose uniformly among the existing nodes. For the second network, choose according to the current number of links of each existing node. After having 4039 nodes (and 4038 edges), complete 88234 edges choosing randomly two nodes to connect, according to the rules we have defined for each network. Compute the degree distribution of your two networks and compare them.

3. Natural networks

   You have a file with the facebook links among 4039 students of some University Department. Compute the degree distribution of this social network and compare with the two synthetic networks you obtained before.

4. Conclude