Theory of Complex Systems

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Fractals from Dynamical Systems

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Introduction to Complex Systems

From reductionism, mechanism, determinism, and predictability to self-organization, emergence, evolution and adaptation.

From self-organized systems

Random process, Brownian motion, fractional Brownian motion, Hurst parameter, midpoint displacement, variance-time plot, stochastic self-similarity, second-order strict self-similarity, second-order asymptotic self-similarity, long-range dependence, 1/f noise, heavy tailed distributions, FARIMA, M/G/∞, wavelet analysis of fractal signals, MWM.

From non-linear systems

Fractals in nature, Affine transformations and Iterated function systems, symmetry and growth, Lindenmayer´s systems, formal grammars, computability, decidability, Gödel’s incompleteness theorems, recursion, euclidean measures of fractal objects, self-similar dimension, Hausdorff measure, Hausdorff dimension, box-counting dimension

From iterative algorithms

Fractals
From self-organized systems

From non-linear systems

From random processes

From iterative algorithms

Fractals in nature, Affine transformations and Iterated function systems, symmetry and growth, Lindenmayer´s systems, formal grammars, computability, decidibility, Gödel’s incompleteness theorems, recursion, euclidean measures of fractal objects, self-similar dimension, Hausdorff measure, Hausdorff dimension, box-counting dimension

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Introduction to Complex Systems
So far: (1) Deterministic fractals built through algorithms that capture exact self-similarity at different scales, (2) Random fractals built through algorithms that capture stochastic self-similarity at different scales

```matlab
s = 0;
for k = 1:4
    s = [s, s, s;
         s, ones(3^(k-1)), s;
         s, s, s];
end
imagesc(s);
```

This kind of fractals have found many interesting application in modern technology and sciences.

(e.g. a fractal antenna maximizes its effective size in a wide range of frequency scales, preserving the radiation patterns: It is compact, useful for multiple bands or ultra wide band, with applications in cellular networks, RFID cards, etc.)
load DescargasNeuronales.mat
NP = length(x)-1;
for k=1:7
    m = 64*(2^k);
    Ns = floor(NP/m);
    y = zeros(Ns,1);
    for i=1:Ns
        y(i) = sum(x(1+m*(i-1):m*i))/m;
    end
    v(k) = var(y);
end
m = 64*(2.^(1:7));
p=polyfit(log2(m),log2(v),1);
plot(log2(m),log2(v),'r-')
H = p(1)/2 + 1

Random fractals are pervasive in nature and form useful models in physics, biology, economy, engineering, psychology, etc.
A Simple Game: Choose a point within a triangle and then double the distance from the nearest corner along a straight line. Repeat while the point is within the triangle.
A Simple Game: Choose a point within a triangle and then double the distance from the nearest corner along a straight line. Repeat while the point is within the triangle.

Let’s Play!
How can we know which points to choose in order to win? ¡Choose any interior point and go backwards!!
(Halve the distance to ... ¡a randomly chosen corner! Then, repeat eternally)
The set of points you visit on the long run will be winning points
\begin{verbatim}
NI = 100000;
P = [0 1 1/2; 0 0 \sqrt(3)/2];
x=[0.5; 0.5];
X=zeros(2,NI);
for i=-200:NI
    r=randi(3);
    x=x + (P(:,r)-x)/2;
    if i>0, X(:,i)=x; end
end
line(X(1,:),X(2,:),'linestyle','none','marker','.','markersize',1); axis square
\end{verbatim}
Knowing we should begin in a serpinsky point, we can do (a little) better in the game 😊
$P[1 \text{ paso}] = 1/4$
$P[2 \text{ pasos}] = 3/16$
$P[3 \text{ pasos}] = 9/64$
$P[n \text{ pasos}] = (3/4)^{n-1}/4$
Game of Chaos

NP = 3; NI = 100000; alpha = 1/2;

P=zeros(2,NP);
for i=1:NP
    t=text(.01,.97,[ 'Seleccione el punto ' int2str(i) '.']);
    p=ginput(1); P(:,i)=p(:);
    delete(t); text(p(1),p(2),int2str(i));
end

t=text(.01,.97,'Calculando...'); drawnow
x=[0.5; 0.5];
X=zeros(2,NI);
for i=-200:NI
    r=ceil(NP*rand);
    x=x+alpha*(P(:,r)-x);
    if i>0, X(:,i)=x; end
end
delete(t); line(X(1,:),X(2,:),'linestyle','none','marker','.','markersize',1); axis square

% Otro ejemplo:
% NP = 5; NI = 100000; alpha = 3/5
Game of Chaos

\[ x_n = \frac{1}{2}(Px_i - x_{n-1}), \quad y_n = \frac{1}{2}(Py_i - y_{n-1}) \]

A dynamical system in which the state \((x,y)\) coordinates evolve with time (index \(n\)).

The trajectories are chaotic, but the attractor (the set of visited points as times go to infinity) forms a beautiful fractal.
\[ z_n = c + z_{n-1}^2, \quad z_0 = 0 \]
$z_n = c + z_{n-1}^2, \quad z_0 = 0$
The region where \( \{ z_n = c + z_{n-1}^2, \quad z_0 = 0 \} \) does not escape to \( \infty \) in \( \mathbb{R} \) is \([-2.0 \ 0.25]\).

What is the region where \( \{ z_n = c + z_{n-1}^2, \quad z_0 = 0 \} \) does not escape to \( \infty \) in \( \mathbb{C} \)?

Let us repeat the previous process: choose 1’440.000 values for \( c \) in the complex plane and let us color \( c \) with color 1 if \( |Z_{256}|=0 \), with color 0 if \( |Z_{256}|=\infty \) and other colors in \((0,1)\) for intermediate values of \( |Z_{256}| \).

```matlab
x = linspace(-1.5,0.5,1200);
y = linspace(-1,1,1200);
[X,Y] = meshgrid(x,y);
Z = zeros(1200);
C = X+i*Y;
for k = 1:256
    Z = Z.^2+C;
end
colormap winter(256);
pcolor(x,y,exp(-abs(Z)));
shading flat;
axis('square','equal','off');
```
Pathological monsters!
Cried the terrified mathematician
Every one of them is a splinter in my eye
I hate the Peano Space and the Koch Curve
I fear the Cantor Ternary Set
And the Sierpinski Gasket makes me want to cry
And a million miles away a butterfly flapped its wings
On a cold November day
A man named Benoit Mandelbrot was born
His disdain for pure mathematics
And his unique geometrical insights
Left him well equipped to face those demons down
He saw that infinite complexity
Could be described by simple rules
He used his giant brain to turn the game around
And he looked below the storm
And saw a vision in his head, A bulbous pointy form
He picked his pencil up and he wrote his secret down

Just take a point called \( Z \) in the complex plane
Let \( Z_1 \) be \( Z \) squared plus \( C \)
And \( Z_2 \) is \( Z_1 \) squared plus \( C \)
And \( Z_3 \) is \( Z_2 \) squared plus \( C \) and so on
If the series of \( Z \)'s should always stay close to \( Z \)
And never trend away
That point is in the Mandelbrot Set

Mandelbrot Set you're a Rorschach Test on fire
You're a day-glo pterodactyl
You're a heart-shaped box of springs and wire
You're one badass fucking fractal
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
And you're just in time to save the day
Sweeping all our fears away
You can change the world in a tiny way
Go on change the world in a tiny way
Come on change the world in a tiny way

Mandelbrot Set
by Jonathan Coulton
The error is obvious:

Just take a point called $Z$ in the complex plane
Let $Z_1$ be $Z$ squared plus $C$
And $Z_2$ is $Z_1$ squared plus $C$
And $Z_3$ is $Z_2$ squared plus $C$ and so on
If the series of $Z$'s should always stay close to $Z$
And never trend away
That point is in the Mandelbrot Set

What he should say:

Just take a point called $C$ in the complex plane
Let $Z_1$ be Zero squared plus $C$
And $Z_2$ is $Z_1$ squared plus $C$
And $Z_3$ is $Z_2$ squared plus $C$ and so on
If the series of $Z$'s should always stay close to 0
And never trend away
The point $C$ is in the Mandelbrot Set
Jonathan described a filled-in Julia set:

Given a complex number $C$
Just take a point called $Z_0$ in the complex plane
Let $Z_1$ be $Z_0$ squared plus $C$
And $Z_2$ is $Z_1$ squared plus $C$
And $Z_3$ is $Z_2$ squared plus $C$ and so on
If the series of $Z$'s should always stay close to $Z_0$
And never trend away
Then the point $Z_0$ is in the filled-in Julia Set for $C$

c = 0.27334 - 0.00742*i;
x=linspace(-1.5,1.5,800);
y=linspace(-1.5,1.5,800);
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
for k=1:60
    Z=Z.^2+c;
end
colormap copper(256)
pcolor(x,y,exp(-abs(Z)));
shading flat;
axis('square','equal','off');
Indeed, the formal definition for the Mandelbrot set is:

Set of parameters $c$ for which the corresponding Julia set is connected
Formally:

Let \( f : \mathbb{C} \to \mathbb{C} \) be a polynomial function of degree \( n \geq 2 \)

Let \( f^k(w) \) be the \( k^{\text{th}} \) iterated composition of \( f \), \( f^k(w) = f(f(...(f(w))...)) \)

Julia sets are defined in terms of the behavior of the iterations \( f^k(z) \) for big values of \( k \)

Let us first define the "filled-in" Julia set of the polynomial \( f \)

\[
K(f) = \left\{ z \in \mathbb{C} : \lim_{k \to \infty} |f^k(z)| < \infty \right\}
\]

Now we can define the Julia set for \( f \) as the border of \( K(f) \):

\[
J(f) = \partial K(f) = \left\{ z \in \mathbb{C} : \forall \varepsilon \in \mathbb{R} \exists w, v \in \mathbb{C} \text{ s.t. } |z - w| < \varepsilon, \ |z - v| < \varepsilon, \lim_{k \to \infty} |f^k(w)| = \infty \text{ and } \lim_{k \to \infty} |f^k(v)| < \infty \right\}
\]

Tipically: \( f(z) = c + z^2 \)
For example, \( c=0: \)

\[
f(z) = z^2 \quad \Rightarrow \quad f^k(z) = z^{2^k} \rightarrow \begin{cases} 
0 & |z_0| < 1 \\
\infty & |z_0| = 1 \\
|z_0| > 1 
\end{cases}
\]

\( f^k(z) \) in the circle \( |z| = 1 \)

\(!This is not a fractal! 😞\)

If we increase \( c \) a little bit, \( c=0.3 + 0.3j \), it is still true that \( f^k(z) \rightarrow \infty \) for small \( z \), and \( f^k(z) \rightarrow \infty \) for big \( z \). The Julia set for this value of \( c \) is the border between these two behaviors.

\(!Now this is a fractal! 😊\)
A fast method for drawing the border

\[ J(c) \text{ is invariant, forward and backwards, with respect to } f(z) = z^2 + c \]

\[ J(c) = f(J(c)) = f^{-1}(J(c)) \]

Si \( z \in J(c), \quad \pm \sqrt{z - c} \in J(c) \)
A faster way to plot the Mandelbrot set:
Color each pixel according to the number of iterations for escaping to \( \infty \)

```matlab
x = linspace(-1.5,0.6,1000);
y = linspace(-1,1,1000);
[X,Y] = meshgrid(x,y);
Z = zeros(1000);
C = X+i*Y;
for i = 1:1000
    for j=1:1000
        z = 0;
        for t=1:256
            z = z.^2+C(i,j);
            if abs(z)>2, break; end
        end
        Z(i,j)=t-1;
    end
end
pcolor(x,y,Z);

[For coloring : shading flat; axis('square','equal','off');cm = zeros(256,3); cm(1:40,1) = (0:39)/39; colormap(cm); brighten(0.5)]
```
How many iterations to reach $|z| > 2$?

David Ball, 1991
How many iterations to reach \( |z| > 2 \)?

\[
\text{im} = 1; \\
\text{epsilon} = \text{zeros}(7,1); \\
\text{iteraciones} = \text{zeros}(7,1); \\
\text{for } k = 1:7 \\
\quad \text{im} = \text{im}/10; \\
\quad c = -0.75 + \text{im} \times 1j; \\
\quad z = 0; \\
\quad \text{for } t=1:100000000 \\
\quad \quad z = z \times z + c; \\
\quad \quad \text{if } \text{abs}(z) > 2, \text{break}; \text{end} \\
\text{end} \\
\text{epsilon}(k) = \text{im}; \\
\text{iteraciones}(k) = t; \\
\text{end}
\]

<table>
<thead>
<tr>
<th>Epsilon</th>
<th>Iteraciones</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1000000</td>
<td>33</td>
</tr>
<tr>
<td>0.0100000</td>
<td>315</td>
</tr>
<tr>
<td>0.0010000</td>
<td>3143</td>
</tr>
<tr>
<td>0.0001000</td>
<td>31417</td>
</tr>
<tr>
<td>0.0000100</td>
<td>314160</td>
</tr>
<tr>
<td>0.0000010</td>
<td>3141593</td>
</tr>
<tr>
<td>0.0000001</td>
<td>31415927</td>
</tr>
</tbody>
</table>

What the heck is \( \pi \) doing, hidden within the Mandelbrot set?

How many more secrets are hiding within it?
The Feigenbaum constant 4.669202…
As fundamental as $e$, $\pi$, $i$, 0 or 1.
Dynamical Systems

A dynamical system changes with time:

- Normally, what changes is the position
- Or the velocity
- Or acceleration… Movement!
Dynamical Systems

Sometimes, other states change:
- Reactant rate
- Temperature
- Pressure
- Viscosity
- Mass
- Friendship
- Social tension
- Commercial flows

- Mood
- Peace
Dynamical Systems

- Along with what changes, it is important to know how does it change
- Changing rules
  - Mathematical models

\[
\frac{d}{dt} x(t) = f(x(t), u(t), t), \quad x(0) = x_0 \\
y(t) = g(x(t), u(t), t)
\]
What a dynamical system can do?

• Maybe reaches an equilibrium point

• Maybe repeats a cycle periodically

• Maybe more strange things...
Linear dynamical systems

\[ F(t) \rightarrow v(t) \rightarrow \rho \cdot v(t) \]

Newton: \[ \frac{M}{\rho} \frac{d}{dt} v(t) + v(t) = \frac{1}{\rho} F(t) \]

There is only one equilibrium point (the origin \( F(t) = \rho v(t) = 0 \)) and, from where ever it is, tends to the origin exponentially.
Linear dynamical systems

\[ \frac{d}{dt} x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \]

\[ y(t) = Cx(t) + Du(t) \]

Positive Real Eigenvalues

Negative Real Eigenvalues

Purely Imaginary Eigenvalues

Complex Eigenvalues with positive real part

Complex Eigenvalues with negative real part
Non-linear dynamical systems

\[ h \frac{d^2}{dt^2} z(t) = u(t) - r \sin(z(t)) \]

- More than one equilibrium point
- Limit cycles
- Bifurcations
- Synchronization
- Sensitivity to initial conditions
- etc.
Linear and non-linear systems

- Linear systems can be separated into parts, solve each part independently, and recombine the partial solutions to obtain the whole response: **Superposition principle**... A linear system is exactly equal to the sum of its parts.

- In nature, many dynamical systems are not like that: The parts of the system interfere among them (cooperate or compete), so the superposition principle fails dramatically... In non-linear systems, the whole is bigger than the sum of its parts.

Mechanistic Reductionism

Complexity (at least, possibility of complexity)
The logic of linearity

If 40 Amish farmers can build a barn in 8 hours...

...then 1280 of them can do it in 15 minutes.
Some basic concepts

\[ \frac{d}{dt} x(t) = f\left(x(t), u(t), t\right), \quad x(0) = x_0 \]
\[ y(t) = g\left(x(t), u(t), t\right) \]

The simplest system:

\[ \frac{d}{dt} x(t) = f\left(x(t)\right), \quad x(0) = x_0 \]

- Analytic treatment
- Geometric treatment

e.g.

\[ \frac{d}{dt} x(t) = \sin\left(x(t)\right), \quad x(0) = x_0 \]

One of the few non-linear systems that can be easily solved analytically

\[ dt = \frac{dx}{\sin(x)} \Rightarrow t = \int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C \]

And, since \( x(0) = x_0 \),

\[ t = \ln \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right| \]

Exact analytic solutions, but...

What does this mean? Who can understand this?

What will happen to \( x(t), t \to \infty \) if \( x_0 = \pi/4 \)? What will happen to \( x(t), t \to \infty \) for any \( x_0 \)?
Better use graphic methods

\[ t \text{ is the time, } x_1 = x(t) \text{ is the position of a particle and } x_2 = \frac{dx(t)}{dt} \text{ is its velocity} \]

If there is a particle moving on the \( x \) axis with a velocity \( \frac{dx}{dt} = \sin(x) \), the movement will be to the left or to the right depending on whether \( \frac{dx}{dt} \) is lower or greater than zero. Those points with \( \frac{dx}{dt} = 0 \) are **fixed points**, some of them **stable** (attractors) and some of them **unstable** (repellers).
¿What happen to \( x(t) \), \( t \to \infty \) when \( x_0 = \pi/4 \)?

\( \pi/4 \) : velocity is positive and increasing, so \( x(t) \) increases convexly up to \( \pi/2 \).
\( \pi/2 \) : velocity is still positive, but decreasing, so \( x(t) \) increases concavely towards \( \pi \).
\( \pi \) : As \( x(t) \) gets closer to \( \pi \), velocity approaches zero, so \( x(t) \) tends to \( \pi \) asymptotically.

¿What happen to \( x(t) \), \( t \to \infty \) for any \( x_0 \)?

Although it won´t be easy to put numbers in the time axis, we can answer every question qualitatively.
Example: Analyze the system \( \frac{dx}{dt} = x^2 + c \), with \( c = -1 \)

Fixed points satisfy \( x^2 = 1 \)
\( x^* = \pm 1 \)

Stability of fixed points is determined from the vector field:

-1 is stable, +1 is unstable
Example: Analyze the system $\frac{dx}{dt} = x^2 + c$, con $c = -1/2$

Fixed points satisfy $x^2 = \frac{1}{2}$

$x^* = \pm \sqrt{1/2}$

$-\sqrt{1/2}$ is stable, $+\sqrt{1/2}$ is unstable
Example: Analyze the system \( \frac{dx}{dt} = x^2 + c \), con \( c = 0 \)

The fixed point satisfies \( x^2 = 0 \)

0 is stable from the left
0 is unstable from the right
Example: Analyze the system \( \frac{dx}{dt} = x^{2} + c \), con \( c = 1 \)

Fixed points satisfy \( x^{2} = -1 \)

There are no fixed points in the real axis
Bifurcation

parámetro c

Puntos Fijos

Equilibrio estable

Equilibrio inestable
In discrete time

\[ \frac{d}{dt} x(t) = f(x(t), u(t), t), \quad x(0) = x_0 \]
\[ y(t) = g(x(t), u(t), t) \]

\[ x_n = f(x_{n-1}, u_n, n), \quad x_0 \text{ dado} \]
\[ y_n = g(x_n, u_n, n) \]

Equilibrium points

\[ f(x^*) = 0 \]
\[ x^* = f(x^*) \]

for example

\[ x_n = \lambda x_{n-1} \]

It is a linear system, the only equilibrium point is the origin (of course!)
- Unless \( \lambda = 1 \), in which case any initial point is an equilibrium point.

How do we know whether it is stable?
\[ x_n = \lambda x_{n-1}, \quad 0 < \lambda < 1 \quad \Rightarrow \text{The origin is stable} \]

\[ x_n = \lambda x_{n-1}, \quad \lambda > 1 \quad \Rightarrow \text{The origin is unstable} \]
$x_n = x_{n-1}, \quad -1 < \lambda < 0$

$x_n = \lambda x_{n-1}, \quad \lambda < -1$
The diagram illustrates the relation:

\[ x_n = \lambda x_{n-1}, \lambda = 1 \]

And another relation:

\[ x_n = \lambda x_{n-1}, \lambda = -1 \]
As we mentioned, linear systems have very few possible behaviors…
Of course, we knew this:

\[
\begin{align*}
    x_1 &= \lambda x_0 \\
    x_2 &= \lambda x_1 = \lambda^2 x_0 \\
    x_3 &= \lambda x_2 = \lambda^3 x_0 \\
    &\vdots \\
    x_n &= \lambda^n x_0
\end{align*}
\]

But it suggests analysis techniques!

… for example, linearize in an infinitesimal interval around an equilibrium point and see the slope of the tangent:

Find equilibrium points \( x^* = f(x^*) \)

For each one, compute \( m = \frac{d}{dx} f(x^*) \)

Determine the local stability of each point:

\[
\begin{align*}
    m &< -1, \quad m = -1, \quad -1 < m < 0 \\
    m &= 0, \quad 0 < m < 1, \quad m = 1, \quad m > 1
\end{align*}
\]
The Logistic map

- Model for population growing
- The greater the number of individuals in a generation, the greater the reproduction rate
  \[ N_n = R \cdot N_{n-1} \]
- The greater the number of individuals in a generation, the smaller the reproduction rate due to resource scarcity
  \[ R_n = R(N_{n-1}), \text{ p.ej. } R_n = \lambda - bN_{n-1} \]
- Let us define \( x_n = \frac{b}{\lambda} N_n \) as a “normalized population”
  \[ x_n = \lambda x_{n-1} \left( 1 - x_{n-1} \right), \quad 0 \leq x_n \leq 1 \]
Let’s see....

- **Fixed points**

\[ x^* = \begin{cases} 0 \\ \frac{\lambda - 1}{\lambda} \end{cases} \]

- **Derivative of the function at the fixed points**

\[
m = \frac{d}{dx} \left[ \lambda x (1-x) \right] = \lambda (1-2x) \bigg|_{x=\frac{\lambda-1}{\lambda}} = 2 - \lambda
\]

- **Local stability**

\[
m < -1 \quad m = -1 \quad -1 < m < 0 \quad m = 0 \quad 0 < m < 1 \quad m = 1 \quad m > 1
\]

\[
\lambda > 3 \quad \lambda = 3 \quad 2 < \lambda < 3 \quad \lambda = 2 \quad 1 < \lambda < 2 \quad \lambda = 1 \quad \lambda < 1
\]
A non-linear dynamical system

\[ x_{n+1} = \lambda x_n (1 - x_n) \]: If \( \lambda \leq 4 \) and \( x_0 \in [0, 1] \), then the trajectory will remain in the interval \([0, 1]\).

With \( \lambda < 1 \), the trajectory goes to zero.

With \( 1 \leq \lambda \leq 3 \), the trajectory goes to \( 1 - 1/\lambda \).

With \( 3 < \lambda \leq 1 + \sqrt{6} \), the trajectory goes to a period-2 cycle.

Doubling period bifurcation

\[
\begin{align*}
3 & \quad 1 \\
3.449 & \quad 2 \\
3.544 & \quad 4 \\
3.564 & \quad 8 \\
3.568 & \quad 16 \\
3.569 & \quad 32 \\
\vdots & \\
3.570 & \quad \infty
\end{align*}
\]
$\lambda = 1.00$

$x_{n+1}$

$x_n$
with $3.57 < \lambda < 3.829$, there are periodic orbits with every period of the form $2^n$… And also aperiodic orbits!

With $\lambda = 3.829$ appears for the first time a period-3 orbit, which bifurcates to periods 6, 12, 24,… From 3.829 to 4.0 there are periodic orbits of every possible period and aperiodic orbits.
If the scale is hidden, we could not distinguish the three portions of the diagram... The bifurcation diagram is a fractal object.
As always, a very simple program

```matlab
x = zeros(10000,1);  % Calcula 10000 puntos
for L = 0.001:0.001:3.999  % Para distintos parámetros
    x(1) = rand(1);   % iniciando en un punto al azar
    for i=2:10000
        x(i) = L*x(i-1)*(1-x(i-1));
    end  % Pero sólo grafica los últimos 9000
y = unique(x(1001:10000));
plot(L*ones(size(y)),y,'b.','MarkerSize',1)
hold on
end
```
Bifurcation and fundamental laws of nature

- Bifurcation to a periodic orbit of period 2: $\lambda_1 = 3$
- Bifurcation to a periodic orbit of period 4: $\lambda_2 = 1 + \sqrt{6}$
- ...
- Let $\lambda_n$ be the parameter that produces a bifurcation to a limit cycle of period $2^n$.
- The distance between two successive bifurcation points is $\lambda_n - \lambda_{n-1}$
- Consider the sequence $d_k = \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k}$, $k \geq 2$, which represents how fast to the next bifurcation
- $d_k \rightarrow 4.669202$ for the logistic map... And for every map with a single maximum! A universal constant!
$x_{n+1} = \lambda x_n (1 - x_n)$ \quad \lambda \in [1, 4] \quad \text{Bifurcation points}

$z_{n+1} = z_n^2 + c$ \quad c \in [-2, 0.25] \quad \text{New circular-shaped bulb} \quad \Rightarrow \quad c_k = \frac{\lambda_k}{2} \left(1 - \frac{\lambda_k}{2}\right)$
\[ f(x) = \lambda x(1-x), \quad f^{<m>}(x) = f(f(\cdots f(x)))) \text{ } m \text{ times} \]
1 paso, \( \Lambda = 4 \)

2 paso, \( \Lambda = 4 \)

3 paso, \( \Lambda = 4 \)

4 paso, \( \Lambda = 4 \)

5 paso, \( \Lambda = 4 \)

6 paso, \( \Lambda = 4 \)
Chaos Predictability

- Given $x_t$, we only want to know whether, after $m$ steps, the value $x_{t+m}$ exceeds or not $\frac{1}{2}$. The answer is “yes” (1) or “no” (0), so we must provide only one bit of information.

- For $m = 1$, it is enough to know whether $x_t$ falls in one of three regions

- For $m = 2$, it is enough to know whether $x_t$ falls in one of five regions

- For $m = 6$, we need to tell $2^6 + 1 = 65$ possible sections where $x_t$ can fall.

- To predict whether $x_{t+m}$ will exceed $\frac{1}{2}$ we must know $x_t$ with, at least, $m+1$ precision bits

A 64 bit computer will tell us if $x_{t+m}$ will exceed $\frac{1}{2}$ only up to 63 steps in the future!
Only two additional problems

• Let us put all computer memory to represent $x_t$ with high precision
  – Ok. We can predict farther in time. But having still a finite memory, there will always be a value of $m$ from which we cannot say anything.
  – Even worse… ¿can we measure $x_t$ with such precision? Any infinitesimal error will become definitive at some time.
Chaos predictability

The predictability of a coin tossing sequence is null: Most we can say is that we expect that, in a long sequence, almost half of the outcomes will be 0 and half of them will be 1: A purely statistical description (randomness).

The predictability of a chaotic system is high in short periods of time, but not in the long run.
Is there any point in conducting computer simulations?

• **Yes! The shadowing lemma**

• A sequence \( \{x_i, i=1,2,...,N\} \) is a **pseudo-orbit**(\( \varepsilon \)) of a map \( f(\cdot) \) if

\[
\left\{ d \left( x_{i+1}, f(x_i) \right) < \varepsilon, \quad i = 1,2,...,N - 1 \right\}
\]

• A point \( y \) is a **shadow**(\( \delta \)) of the sequence \( \{x_i, i=1,2,...,N\} \) if

\[
\left\{ d \left( x_i, f^{<i>} (y) \right) < \delta, \quad i = 1,2,...,N \right\}
\]
Is there any point in conducting computer simulations?

• Yes! The shadowing lemma

The shadowing lemma

In any region \( \Lambda \), for each \( \delta > 0 \) there exists an \( \epsilon > 0 \) such that every pseudo-orbit\( \epsilon \) \( \{x_i, i=a,a+1,...,b\} \) of \( f \) in \( \Lambda \) has a shadow(\( \delta \)) \( y \) in \( \Lambda \).

1. If your weather forecast was a shame, do not worry: It could happen as you said.
2. Computer simulations cannot predict anything, but they can characterize a chaotic system behavior.
Chaos Characteristics

• Deterministic
  – Future is completely determined from the history of the system, although we cannot compute it

• Sensitive
  – Any perturbation, no matter how small, will change the evolution of the system for ever

• Ergodic
  – A chaotic system will always return to a local region close to its initial state.

• Embedded
  – Chaotic attractors are embedded with an infinite number of unstable periodic orbits.
More than one dimension (e.g., 2)

- Hénon Map

\[ x_{n+1} = a - x_n^2 + by_n \]
\[ y_{n+1} = x_n \]

\( a = 1.29 \)
\( b = 0.3 \) for example

\[ x^* = a - x^*^2 + bx^* \quad \Rightarrow \quad x^* = \frac{1}{2} \left( b - 1 + \sqrt{(b-1)^2 + 4a} \right) = 0.838486 \ldots \]
Hénon Map

```matlab
x = zeros(100000,1);
a = 1.29; b = 0.3;
x(1:2) = (2*rand(2,1)-1)/1.1;
for i=3:100000
    x(i) = a - x(i-1)^2 + b*x(i-2);
end
subplot(211); plot(x(100:200)); subplot(212);
plot(x(10:99999),x(11:100000),'.','MarkerSize',0.5)
```

Strange attractor:

We can zoom on it indefinitely and will never stop the appearance of more and more details
$b=0.3, \ a \in [1.0, 1.42]$
Again, IFS for fractals generation:

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} \mapsto
\begin{bmatrix}
a - x^2 + by \\
x
\end{bmatrix}
\]
¿Is network traffic the product of a chaotic dynamical system?

Diagrama de fase de los tiempos entre llegadas

Real traffic over an ethernet LAN

Poisson Traffic
Convection currents
Convection currents on earth mantle produce tectonic displacement
Deterministic Nonperiodic Flow

Edward N. Lorenz

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(Manuscript received 18 November 1962, in revised form 7 January 1963)

Abstract

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions. A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

Foundation of chaos theory

Lorenz did not have a title for a lecture to be given in 1972 at the 139th congress of the American Association for the Advancement of Science. Philip Merilees, the chairperson, presented it as: “Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?”.
Lorenz Equations

\[
\begin{align*}
\frac{dx}{dt} &= ay - ax \\
\frac{dy}{dt} &= bx - y - zx \\
\frac{dz}{dt} &= xy - cz
\end{align*}
\]

- \( t \) : time
- \( x(t) \) : position
- \( y(t) \) : temperature
- \( z(t) \) : “distortion of T”
- \( a, b, c \) : Parameters
Computer solutions to differential equations

\[ \dot{x} = f(x), \quad x(0) = x_0 \]

We ride on the phase point, flowing according to the vector field.

If we begin at \( t=0 \) in \( x(0)=x_0 \) with a velocity \( f(x_0) \), after a very small period of time \( \Delta t \) we would advance a distance \( f(x_0) \Delta t \) so now we will be at \( x(\Delta t)=x_1 \)

\[ x_1 = x_0 + f(x_0) \Delta t \]

And, after several iterations,

\[ x_{n+1} = x_n + f(x_n) \Delta t \]

\textbf{Euler Method}
\[ \frac{d}{dt} x(t) = \sin(x(t)), \quad x(0) = x_0 \] through Euler

\[ x_0 = \pi \cdot [-2 -1.9 -3/2 -1 -1/2 -0.1 0 0.1 1/4 1/2 1 3/2 1.9 2]; \]

for i=1:14
    x(1) = x0(i);
    for j=2:10000
        x(j) = x(j-1) + 0.0004*\sin(x(j-1));
    end
    plot(0.0004*(0:9999),x)
end

\[ x_2 \]

\[ x_1 \]

-2\pi -\pi 0 \pi 2\pi
\[
\frac{d}{dt} x(t) = \sin(x(t)), \quad x(0) = x_0
\]

通过 Euler

\[x(1) = 0.01; \quad y(1) = 0.01;\]

for \( j=2:20000 \)

if \( j==200*\text{floor}(j/200) \)

\[y(1+j/200) = y(j/200) + 0.08*\sin(y(j/200));\]

end

\[x(j) = x(j-1) + 0.0004*\sin(x(j-1));\]

end

plot(0.0004*(0:19999),x,'b-',0.08*(0:100),y,'r--')

- 只是估计的在左端点的导数
- 更好是使用在间隔内导数的平均值

\[
d = x_n + f(x_n)\Delta t
\]

\[
x_{n+1} = x_n + \frac{1}{2}\left(f(x_n) + f(d)\right)\Delta t
\]
\[ \frac{dx(t)}{dt} = \sin(x(t)), \quad x(0) = x_0 \quad \text{Through 2}\textsuperscript{nd} \text{order Euler} \]

\[
\begin{align*}
x(1) &= 0.01; \quad y(1) = 0.01; \\
\text{for } j=2:20000 \\
&\quad \text{if } j==200*\text{floor}(j/200) \\
&\quad \quad d = y(j/200) + 0.08\sin(y(j/200)); \\
&\quad \quad y(1+j/200) = y(j/200) + 0.04(\sin(y(j/200)) + \sin(d)); \\
&\quad \text{end} \\
&\quad x(j) = x(j-1) + 0.0004\sin(x(j-1)); \\
\text{end} \\
\text{plot}(0.0004*(0:19999),x,'b-',0.08*(0:100),y,'r--') \\
\text{legend ('20000 pasos', '100 pasos')}
\end{align*}
\]

- First order Euler
- Second order Euler
4th order Runge-Kutta method

\[ k_1 = f(x_n) \Delta t \]
\[ k_2 = f(x_n + k_1 / 2) \Delta t \]
\[ k_3 = f(x_n + k_2 / 2) \Delta t \]
\[ k_4 = f(x_n + k_3) \Delta t \]
\[ x_{n+1} = x_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]
Multidimensional 4\textsuperscript{th} order Runge-Kutta method

\[
\tilde{k}_1 = f(\tilde{x}_n)\Delta t
\]
\[
\tilde{k}_2 = f\left(\tilde{x}_n + \frac{\tilde{k}_1}{2}\right)\Delta t
\]
\[
\tilde{k}_3 = f\left(\tilde{x}_n + \frac{\tilde{k}_2}{2}\right)\Delta t
\]
\[
\tilde{k}_4 = f(\tilde{x}_n + \tilde{k}_3)\Delta t
\]
\[
\tilde{x}_{n+1} = \tilde{x}_n + \frac{1}{6}\left(\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4\right)
\]

\texttt{function} \ x = \text{AtractorDeLorenz}\n\texttt{\hspace{1cm} x(:,1) = [1 2 3]';}\n\texttt{\hspace{1cm} for \ j=2:10000}\n\texttt{\hspace{1cm} \hspace{1cm} k1 = 0.01*ecuacion(x(:,j-1));}\n\texttt{\hspace{1cm} \hspace{1cm} k2 = 0.01*ecuacion(x(:,j-1) + k1/2);}\n\texttt{\hspace{1cm} \hspace{1cm} k3 = 0.01*ecuacion(x(:,j-1) + k2/2);}\n\texttt{\hspace{1cm} \hspace{1cm} k4 = 0.01*ecuacion(x(:,j-1) + k3);}\n\texttt{\hspace{1cm} \hspace{1cm} x(:,j) = x(:,j-1) + (k1 + 2*k2 + 2*k3 + k4)/6;}\n\texttt{\hspace{1cm} end}\n\texttt{\hspace{1cm} x = x';}\n\texttt{\hspace{1cm} plot3(x(:,1),x(:,2),x(:,3));}\n
\texttt{function} \ x = \text{ecuacion}\(x)\n\texttt{\hspace{1cm} x = [10*(x(2)-x(1)); 28*x(1)-x(2)-x(1)*x(3); x(1)*x(2)-(8/3)*x(3)];}\n
\texttt{\hspace{1cm} UNIバiversity Distリal Francisco Jose de Caldas}\n\texttt{\hspace{1cm} Doctorado en Ingenieria}\n\texttt{\hspace{1cm} Theory of Complex Systems}\n\texttt{\hspace{1cm} Marco Aurelio Alzate Monroy}
Time evolution: ¡Chaos!

Something more interesting?
Phase Space

Lorenz Attractor

Note: Matlab already has Runge-Kutta: ☺

\[
g = \text{inline}(['10*(x(2)-x(1))'; '28*x(1)-x(2)-x(1)*x(3)'; 'x(1)*x(2)-(8/3)*x(3)']);
[t,x] = ode45(g,[0:0.01:100],[1;2;3]);
plot3(x(:,1),x(:,2),x(:,3));
\]
A (very) brief introduction to chaos control


\[ x_n = \lambda x_{n-1} (1 - x_{n-1}), \quad 0 \leq x_n \leq 1 \]

- In its natural state, \( \lambda \) corresponds to a chaotic regime, e.g. \( \lambda = 3.78 \)
- For some reason, we cannot move \( \lambda \) to an stable regime
- We can only produce very small perturbations at particular instants of time, where the system remains in a chaotic regime
- We take advantage of the existence of a fixed point (unstable, of course) at \( x^* = (\lambda - 1)/\lambda \)
- When \( x(t) \) gets close to \( x^* \), we adjust \( \lambda \)

```matlab
x(1) = 0.5; % Punto inicial
ro = 3.78; % Parámetro en régimen caótico
r = ro; % Nuestro parámetro
for i=2:2500
    x(i) = r*x(i-1)*(1-x(i-1)); % Evalúa el siguiente estado
    rp = 1/(1-x(i)); % Y, si está muy cerca del punto fijo
    if abs(rp-ro)<0.01 % hace un ligerísimo ajuste al parámetro
        r = rp;
    end
end
plot(x)
```
In the computer, the system will remain at the equilibrium point but, in reality, noise or Imprecision will take it out of control.

\[
x(t) = 0.5;
\]

\[
ro = 3.78;
\]

\[
r(1) = ro;
\]

for i=2:2500

\[
x(i) = r(i-1)*x(i-1)*(1-x(i-1)) + 0.0001*randn(1); \quad \text{Evalúa el siguiente estado}
\]

\[
rp = 1/(1-x(i)); \quad \text{Y, si está muy cerca del punto fijo}
\]

if abs(rp-ro)<0.01

\[
r(i) = rp; \quad \text{hace un ligerísimo ajuste al parámetro}
\]

else

\[
r(i) = r(i-1);
\]

end

end

subplot(211); plot(x)

subplot(212); plot(r,'r')
If noise is too high, the system can get out of control every now and then.
\[
x(1) = 0.5; \\
ro = 3.78; \\
r(1) = ro; \\
x(2) = r(1)*x(1)*(1-x(1)); \\
r(2) = ro \\
for i=3:5000 \\
    rp = x(i-2)/(x(i-1)*(1-x(i-1))); \\
    if abs(rp-ro)<0.01 \\
        r(i) = rp; \\
    else \\
        r(i) = r(i-1); \\
    end \\
x(i) = r(i)*x(i-1)*(1-x(i-1)); \\
end \\
subplot(211); plot(x) \\
subplot(212); plot(r,'r')
\]

- Chaos control in ventricular fibrillation
- Chaos control in brain activity associated with epileptic attack
¡Homework!
Homework #6, part I

• Plot the set of values of $c$ in the complex plane for which the orbit of $z_0=0$ under iteration of the complex function $z_{n+1} = z_n^d + c$ remains bounded, for different values of $d \in \mathbb{R}$
Homework #6, part II

• Consider the tent map:

\[ x_{n+1} = \begin{cases} 
  \lambda x_n & x_n < 1/2 \\
  \lambda (1 - x_n) & x_n \geq 1/2 
\end{cases} \]

• Draw a bifurcation diagram for 1 \leq \lambda \leq 2.

• Write a program to draw the cobweb of the tent map for any given value of \lambda and x_0.

• For \lambda > 2 the bifurcation diagram gets empty: almost all orbits seem to diverge. However, still there could be some bounded unstable orbits. Consider the case \lambda = 3 and find all the bounded orbits, i.e., find all the initial conditions \( x_0 \in [0,1] \) from which the system remains bounded (\( x_n \in [0,1] \ \forall n \in \mathbb{N} \)); this is the Julia set of the tent map for \lambda = 3. Do you recognize some interesting pattern?
Homework #6, part III

- Consider the following 3-dimensional differential equation (Rossler equation):
\[
\frac{dx}{dt} = -(y + z) \\
\frac{dy}{dt} = x + 0.2y \\
\frac{dz}{dt} = 0.2 + z(x - 5.7)
\]

- Use the Runge-Kutta numerical integration method to plot \(x(t)\) vs \(t\), \(y(t)\) vs \(t\) and \(z(t)\) vs \(t\) for two slightly different initial conditions.
- Use the Runge-Kutta numerical integration method to draw a 3-dimensional plot of the trajectory (a strange attractor)
- Discuss the results