Reacting systems with nonlocal diffusion: Bifurcation and patterns

Peter Bates
Michigan State University

Guangyu Zhao
University of the West Indies

Yu Liang
Michigan State University

U. Distrital Francisco José de Caldas, Bogotá July 22, 2017

Thanks to NSF and IMA for support

1. Classical Turing Instability

2. Turing Instability with nonlocal diffusion–Spectral theory

3. Chafee-Infante with nonlocal diffusion and an open problem!

TURING PATTERNS

How do such beautiful patterns arise from a single fertilized egg? How even is symmetry broken?

First, I need to discuss

eigenvalues and eigenfunctions

of differential operators and see what that means for the movement of a diffusing quantity.

You know about eigenvalues and eigenvectors of matrices:

$$M\phi = \lambda\phi.$$

Consider the movement of a spatially distributed heat $u(x, y; t)$ on a region Ω.
For simplicity, take $\Omega = [0, \pi] \times [0, 2\pi]$.
With the notation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

it can be argued that the heat moves through Ω according to

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{(HE)}$$

and if the boundary of Ω is insulated, then

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega.$$

Through "separation of variables" one finds that solutions have the form

$$u(x,y;t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\cos(nx)\cos(my/2)e^{-(n^2+m^2/4)t},$$

which we write

$$u(x,y;t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x,y)e^{-\mu_{n,m}t},$$

or better still

$$u(x,y;t) = \sum_{k=1}^{\infty} \gamma_k\phi_k(x,y)e^{-\lambda_k t} \qquad (Eig - Exp),$$

with $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \ldots$.

Note that

$$\Delta\psi_{n,m}(x,y) = -(n^2 + (m/2)^2)\psi_{n,m}$$

so

$$\frac{\partial\psi_{n,m}}{\partial t} = \Delta\psi_{n,m}(x,y) = -(n^2+(m/2)^2)\psi = -\mu_{n,m}\psi_{n,m}$$

and

$$\psi(x,y;t) = \psi(x,y;0)e^{-\mu_{n,m}t} \to 0 \quad \text{as} \quad t\to\infty.$$

If, instead we consider

$$u_t = \Delta u + \rho u,$$

then the eigenfunctions, $\psi_{n,m}$ remain the same but the eigenvalues, $\mu_{n,m} = \lambda_k$, are increased by ρ and the solution becomes

$$u(x, y; t) = \sum_{k=1}^{\infty} \phi_k(x, y; 0) e^{(\rho - \lambda_k)t}.$$

What happens as $t \to \infty$?

Consider the following system for $d > 1$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u,v), \\ \dfrac{\partial v}{\partial t} = d\Delta v + g(u,v) \qquad \text{in} \quad \Omega \times [0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

We assume that $(p,q)^T \in \mathbb{R}^2$ is a stable equilibrium of the kinetic system, that is,

$$f(p,q) = g(p,q) = 0 \quad \text{and} \quad \mathrm{Jac}(f,g)(p,q)$$

has two eigenvalues with negative real parts.

Normally, we expect diffusion to favor (stabilize) constant states. Consider the linearized R-D system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \begin{pmatrix} f_u(p,q) & f_v(p,q) \\ g_u(p,q) & g_v(p,q) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

Let

$$A = D\Delta + B \tag{3}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad B = \begin{pmatrix} f_u(p,q) & f_v(p,q) \\ g_u(p,q) & g_v(p,q) \end{pmatrix},$$

and find the spectrum of A. We may look at the spectrum of A as it operates on various Fourier modes:
Set

$$A(s) = Ds + B,$$

where s is the spectral parameter from Δ, so $s \leq 0$ is of relevance. Eigenvalues? Instability of some eigenmodes?

$$\det[A(s) - \lambda I] = \lambda^2 - b(s)\lambda + c(s),$$

where $b(s) = (f_u + g_v)|_{(p,q)} + s(1+d)$,
$c(s) = (f_u g_v - f_v g_u)|_{(p,q)} + s(df_u + g_v)|_{(p,q)} + ds^2$.

Eigenvalues for different spatial modes:

$s \to \lambda(s)$ such that $\det A(s) = 0$ has two real branches $\lambda^-(s) < \lambda^+(s)$ for all $s \leq 0$.

One can easily show that

$\lambda^-(s)$ is strictly increasing

and

$\lambda^+(0) < 0$ but $\lambda^+(s)$ has a unique maximum $\lambda^+_{\max}$ attained at some $s_{\max} < 0$.

Is this maximum positive?

i.e., is the $s_{\max}$ mode unstable?

It is if we impose the Turing conditions:

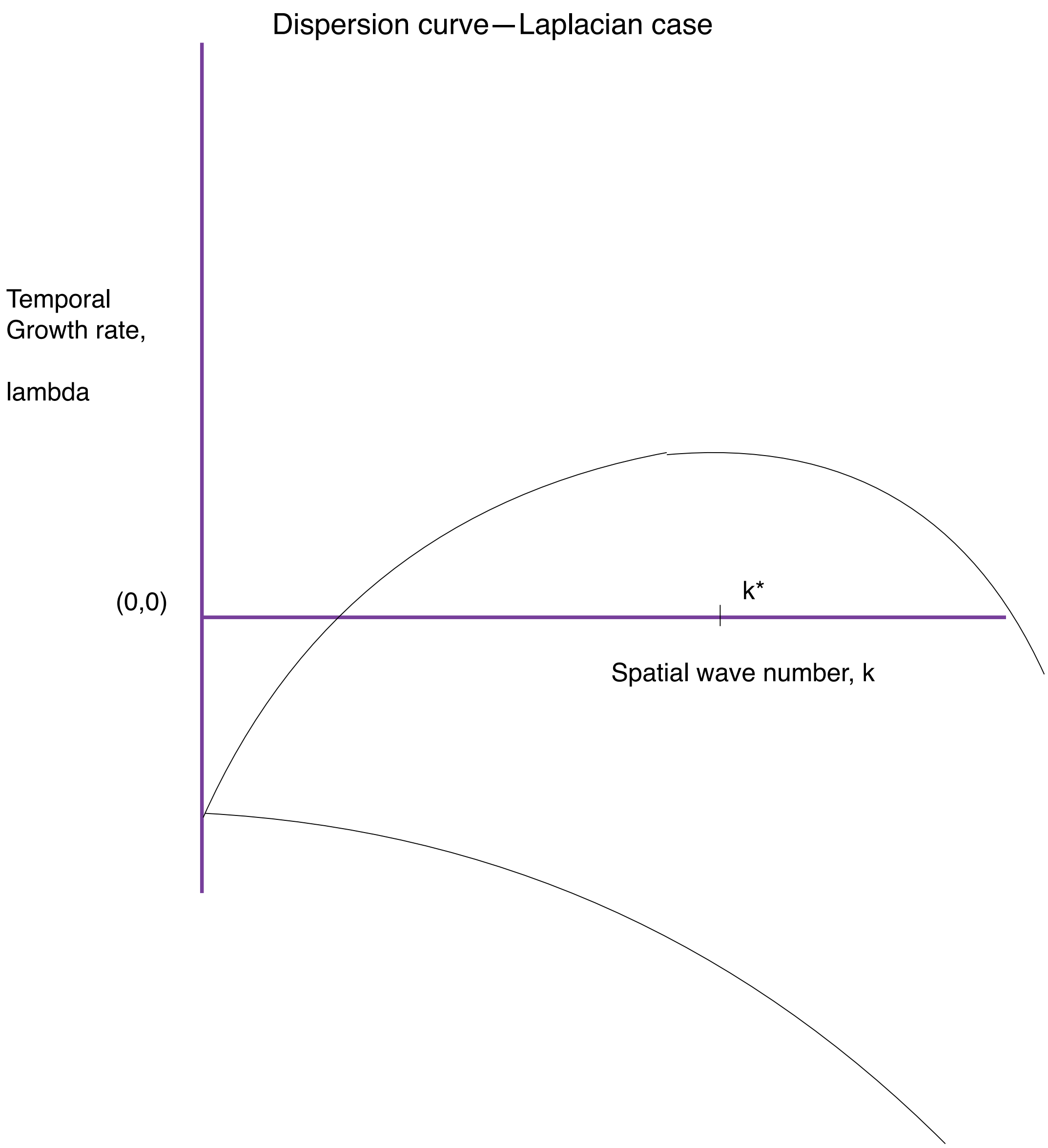
(H1) $f_u|_{(p,q)} > 0,$

(H2) $(f_u + g_v)|_{(p,q)} < 0 < (f_u g_v - f_v g_u)|_{(p,q)},$

(H3) $(f_u + g_v)^2 - 4(f_u g_v - f_v g_u)|_{(p,q)} > 0.$

(H4) $(df_u + g_v)|_{(p,q)} > 0,$

(H5) $(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u)|_{(p,q)} > 0.$


Dispersion curve—Laplacian case
Temporal
Growth rate,
lambda
(0,0)
k*
Spatial wave number, k

What motivated the work I want to describe is interest in higher eigenvalues–

The (Alan) Turning instability $\rightarrow$ emergence of patterns in chemical and biological systems morphogenesis.

We can consider the system

$$\frac{du}{dt} = d_u L u + f(u, v)$$

$$\frac{dv}{dt} = d_v L v + g(u, v)$$

and ask if Turing patterns emerge from a homogeneous steady state which is stable under the kinetics, i.e., that state becomes unstable when 'diffusion' is added

For stability of a homogeneous steady state (p, q) under kinetics: $\det \mathrm{Jac}(f, g)(p, q) > 0, \ \mathrm{Tr}\ \mathrm{Jac}(f, g)(p, q) < 0.$

Adding diffusion one is directed to consider the spectrum of

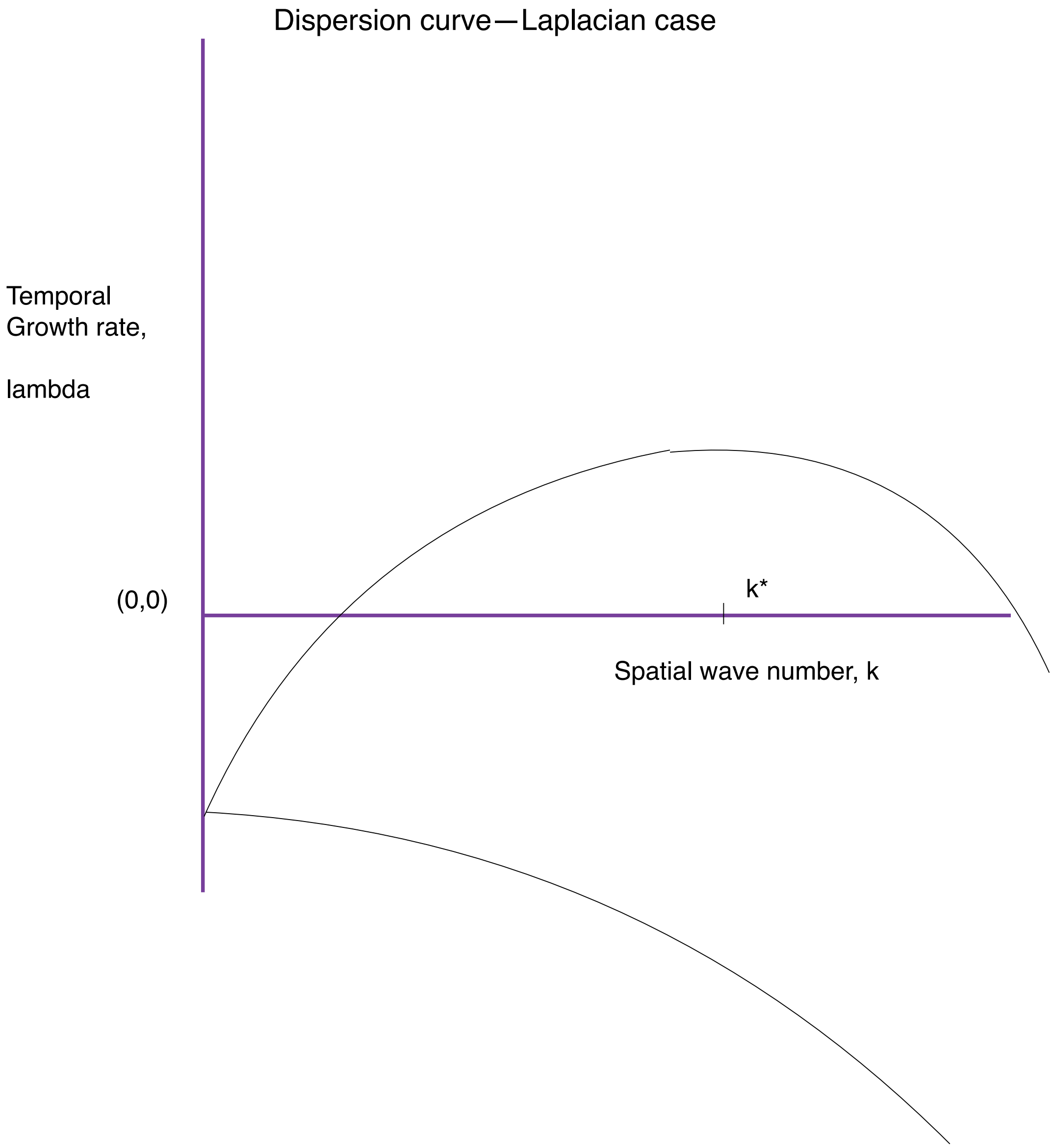
$$\mathrm{diag}(d_u, d_v)L + \mathrm{Jac}(f, g)(p, q).$$

in some *function* space.

First we need $\sigma(L)$, but this is not easy to find for a general kernel and a general domain.

Open problem 1. What is the spectrum of L?

In Turing's case L is the Laplacian



Consider the case $\Omega = \mathbb{R}^n$ with J normalized $\int J = 1$,

$$Lu(x) \equiv \int_{\mathbb{R}^n} J(x-y)u(y)dy - u(x).$$

Now rescale space and consider

$$J_\varepsilon(x) \equiv \varepsilon^{-n} J(x/\varepsilon)$$

and write

$$L_\varepsilon u(x) \equiv \frac{1}{\varepsilon^2}[J_\varepsilon * u - u]$$

then one might imagine that as $\varepsilon \to 0$, $L_\varepsilon \to \Delta$.

Clearly this cannot be true since the difference between the two operators is an unbounded operator. However, even if J changes sign

Lemma (B-Chen-Chmaj, 2003, 2005)
For all $\phi \in H^2(\mathbb{R})$

$$L_\varepsilon \phi \to c_J \Delta\phi \ \text{as } \varepsilon \to 0,$$

where $c_J = \int |z|^2 J(z)/2.$

Similarly, on a bounded domain $\Omega \subset \mathbb{R}^n$ with

$$L_\varepsilon u(x) \equiv \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x-y)(u(y)-u(x))dy$$

(and with J$\geq$ 0) the same result was proved by Cortazar, et al in 2008. In this case Δ is the Neumann Laplacian.

Lemma 1 *(Cortazar, Elgueta, Rossi, Wolanski, ARMA 2008)*
If

$$|(L_{\varepsilon_k} w_k, w_k)| \leq C,$$

where $\|w_k\| = 1$ *and* $\varepsilon_k \to 0$*, then* $\{w_k\}$ *is precompact in* L^2*, a subsequence converges to some* $w \in H^1$ *and*

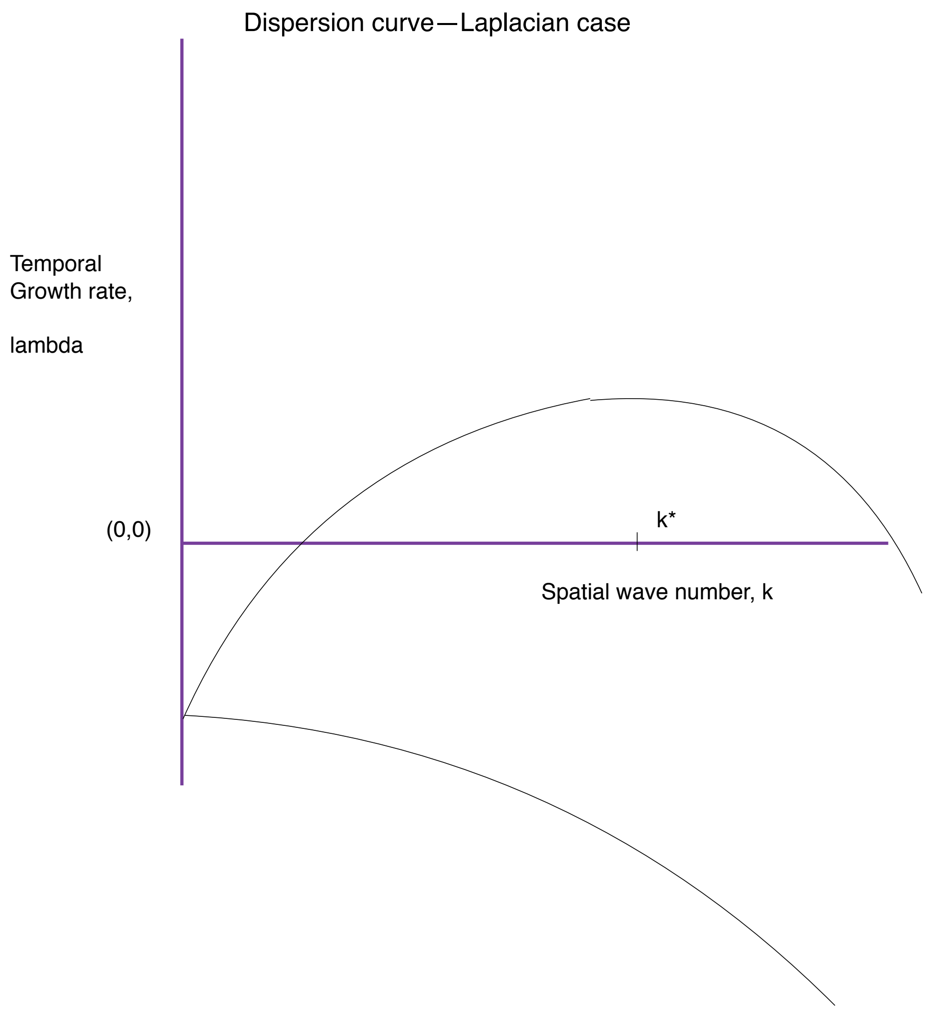
$$L_{\varepsilon_k} w_k \textit{ converges weakly to } c_J \Delta^N w.$$

We want to know about the spectrum of L_ε for Turing instability. Notice that the previous lemmas give a sort of pointwise (not operator) convergence only. Consequently, it is a nontrivial question to ask if the spectrum of L_ε is close to that of $c_J \Delta^N$.

It is clear that $\sigma(L_\varepsilon), \sigma(c_J \Delta^N) \subset (-\infty, 0]$, since both operators are self-adj., 0 is in both spectra, and both operators have numerical range in $(-\infty, 0]$.

Actually, it is a trivial question, since the answer is NO! (A bounded set cannot approximate an unbounded set)

HOWEVER, recall the essentials of the Turing instability:



What we need is for each $M > 0$

$$\sigma(L_\varepsilon) \cap [-M, 0] \to \sigma(c_J\Delta) \cap [-M, 0] \text{ as } \varepsilon \to 0.$$

Lemma 2 *Given a compact subset* $\Theta \subset \rho(c_J\Delta^N)$, *there exists* $\epsilon_\Theta > 0$ *such that* $\Theta \subset \rho(L_\epsilon)$ *if* $\epsilon \le \epsilon_\Theta$.

Proof: Assume there is a sequence $\varepsilon_k \to 0$ and $\lambda_k \in \Theta \cap \sigma(L_{\varepsilon_k})$ (w.l.o.g. $\lambda_k \to \lambda^* \in \Theta$) so that for each k, there is a (Weyl) sequence $\{v_k^j\}_j \subset 1^\perp \cap \{\|v\| = 1\}$ for which

$$\|(L_{\epsilon_k} - \lambda_k I)v_k^j\| \to 0 \text{ as } j \to \infty.$$

Choose $w_k \in 1^\perp$ with $\|w_k\| = 1$ and

$$\|(L_{\epsilon_k} - \lambda_k I)w_k\| \le \frac{1}{k}.$$

Now

$$|(L_{\epsilon_k} w_k, w_k)| \le \|(L_{\epsilon_k} - \lambda_k I)w_k\| + |\lambda_k| \le C.$$

Apply Lemma 1 to get a subsequence $\{w_k\}$ converging to some $w \in H^1$ and from the second part of that lemma we find that w is an eigenfunction for $c_J\Delta^N$ corresponding to $\lambda^* \in \Theta$, a contradiction.

Proposition 0.1 *Assume that* $\Theta \subset \rho(L_\epsilon) \cap \rho(c_J \Delta^N)$ *for all* $\epsilon \leq \epsilon_\Theta$, *where* $\Theta \subset \mathbb{C}$ *is compact. Then there exist* $\theta > 0$ *and* $\bar{\epsilon}_\Theta > 0$ *such that*

$$\|(\lambda I - L_\epsilon)^{-1}\| \leq \theta \quad \textit{for all} \quad \lambda \in \Theta,\ \ \epsilon \leq \bar{\epsilon}_\Theta.$$

Furthermore,

$$(\lambda I - L_\epsilon)^{-1} u \to (\lambda I - c_J \Delta^N)^{-1} u$$

strongly in $L^2(\Omega)$ *as* $\epsilon \to 0$, *for each* $u \in L^2(\Omega)$, *uniformly in* $\lambda \in \Theta$.

Theorem 1 *Assume that* $\mu \in \sigma(c_J \Delta^N)$ *and let* $B_\delta = \{\lambda \in \mathbb{C} : |\lambda - \mu| \leq \delta\}$ *with* $\delta > 0$ *so small that* $B_\delta \cap \sigma(c_J \Delta^N) = \{\mu\}$. *Then there exists* $\epsilon_\delta > 0$ *such that* $B_\delta \cap \sigma(L_\epsilon) \neq \varnothing$ *and* $B_\delta \cap \sigma(L_\epsilon) \subset \sigma_d(L_\epsilon)$ *for all* $\epsilon \leq \epsilon_\delta$. *Furthermore, if dim* $\ker(\mu I - c_J \Delta^N) = m$ *then* $L_\epsilon (\epsilon \leq \epsilon_\delta)$ *has at most* m *isolated eigenvalues* $\mu_j^\epsilon \in B_\delta (1 \leq j \leq m)$ *and the total multiplicity is* m.

The proof uses ideas from Kato, including the contour integral representation of spectral projection operators, the proposition above, and the convergence of the nonlocal operator to the Laplacian, among other things.

TURING PATTERNS

Consider the following system for $d > 1$

$$\begin{cases} \dfrac{\partial u}{\partial t} = L_\varepsilon u + f(u,v), \\ \dfrac{\partial v}{\partial t} = dL_\varepsilon v + g(u,v) \qquad \text{in} \quad \Omega \times [0,\infty). \end{cases} \tag{4}$$

We assume that $(p,q)^T \in \mathbb{R}^2$ is a stable equilibrium of the kinetic system, that is, $f(p,q) = g(p,q) = 0$ and $\mathrm{Jac}(f,g)(p,q)$ has two eigenvalues with negative real parts. Now consider the linearized R-D system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} L_\epsilon u \\ L_\epsilon v \end{pmatrix} + \begin{pmatrix} f_u(p,q) & f_v(p,q) \\ g_u(p,q) & g_v(p,q) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{5}$$

Let

$$A_\epsilon = DL_\epsilon + B \tag{6}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad B = \begin{pmatrix} f_u(p,q) & f_v(p,q) \\ g_u(p,q) & g_v(p,q) \end{pmatrix}.$$

We impose the Turing conditions for local diffusion:

(H1) $f_u|_{(p,q)} > 0, \ \ \mathrm{tr}B = (f_u + g_v)|_{(p,q)} < 0.$

(H2) $\det B = (f_u g_v - f_v g_u)|_{(p,q)} > 0.$

(H3) $(f_u + g_v)^2 - 4(f_u g_v - f_v g_u)|_{(p,q)} > 0.$

(H4) $(df_u + g_v)|_{(p,q)} > 0.$

(H5) $(df_u + g_v)^2 - 4d(f_u g_v - f_v g_u)|_{(p,q)} > 0.$

Set

$$A(s) = B + sD.$$

$$\det[A(s) - \lambda I] = \lambda^2 - b(s)\lambda + c(s),$$

where $b(s) = (f_u + g_v)|_{(p,q)} + s(1 + d)$,
$c(s) = (f_u g_v - f_v g_u)|_{(p,q)} + s(df_u + g_v)|_{(p,q)} + ds^2$.

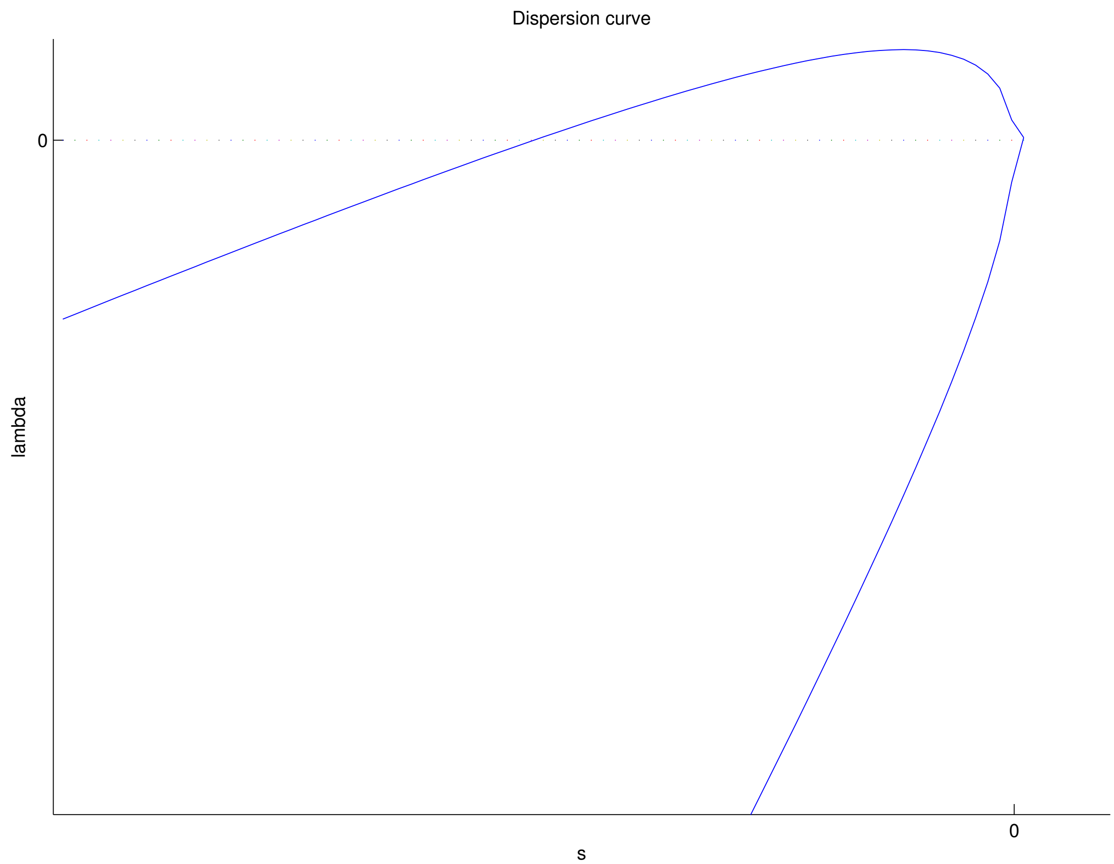
$s \to \lambda(s)$ such that $\det A(s) = 0$ has two real branches $\lambda^-(s) < \lambda^+(s)$ for all $s \leq 0$.

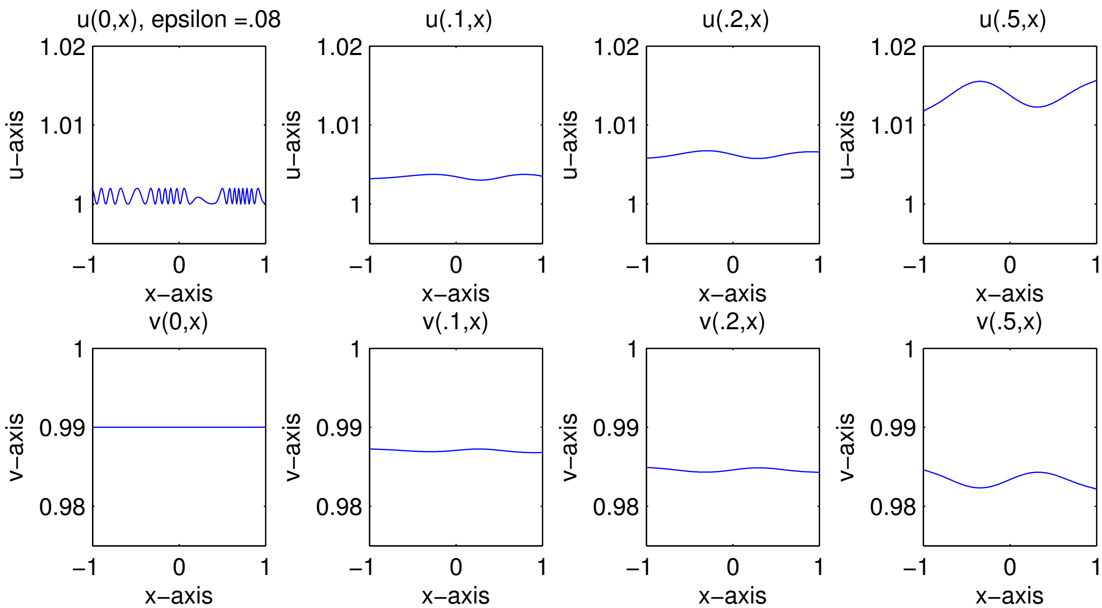
$\lambda^-(s)$ is strictly increasing

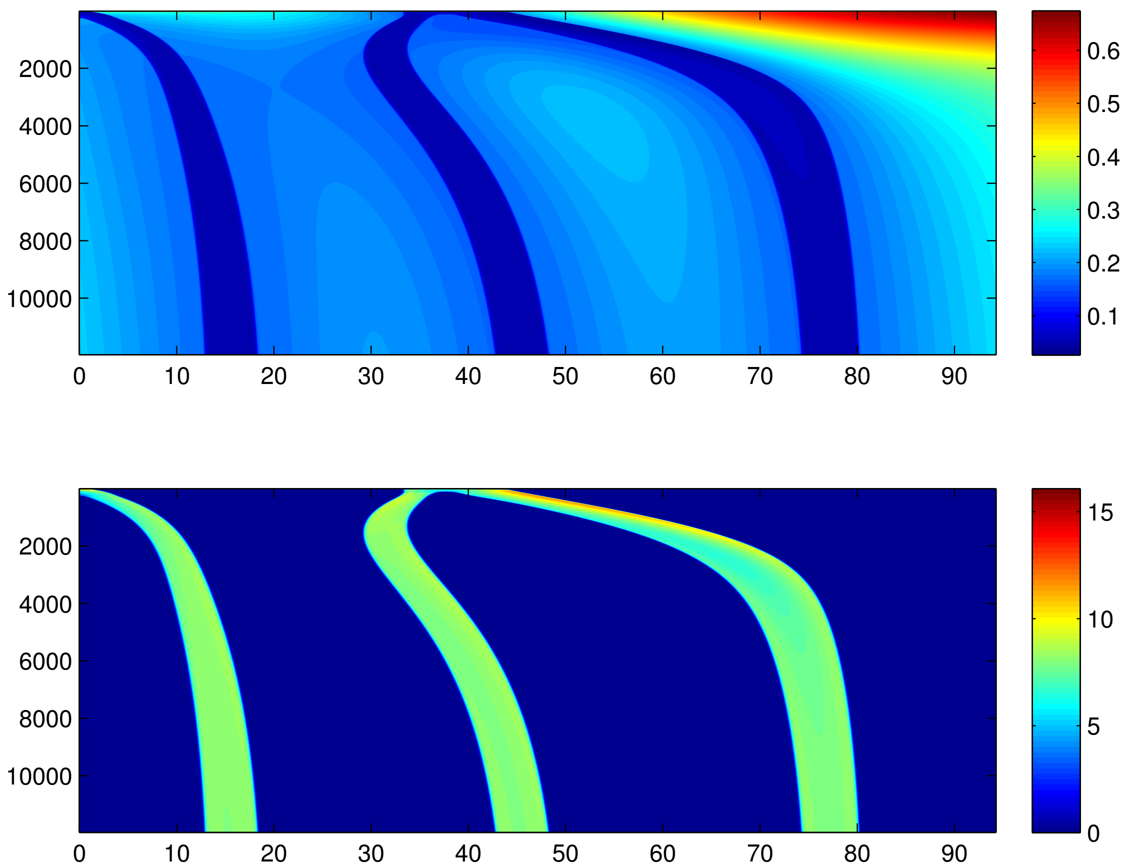
$\lambda^+(0) < 0$, $\lambda^+(s)$ has a unique maximum $\lambda^+_{\max}$ attained at some $s_{\max} < 0$.

Note that s stands for the spectral parameter from L_ε and as with the Laplacian, $s \leq 0$ is what concerns us.

For the Schnakenberg system with nonlocal diffusion:




u(0,x), epsilon =.08
u(.1,x)
u(.2,x)
u(.5,x)
v(0,x)
v(.1,x)
v(.2,x)
v(.5,x)
u−axis
v−axis
x−axis
1.02
1.01
1
1
0.99
0.98
−1
0
1


2000
4000
6000
8000
10000
0 10 20 30 40 50 60 70 80 90
0.6
0.5
0.4
0.3
0.2
0.1
2000
4000
6000
8000
10000
0 10 20 30 40 50 60 70 80 90
15
10
5
0

.

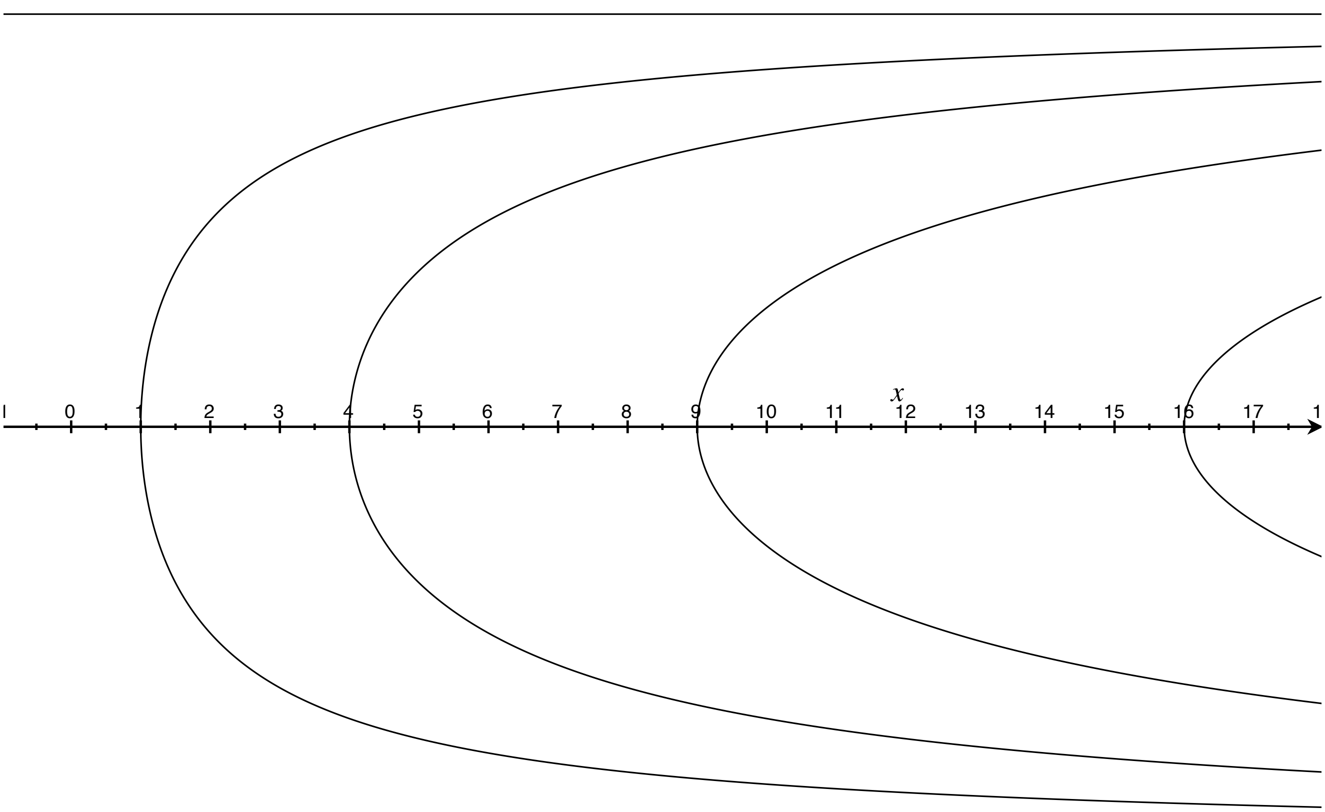
Results with Yu Liang.

$$L_\epsilon u + \lambda u(1-u^2) = 0 \quad \text{on } [0,\pi], \qquad \text{(NCI)}$$

Compare with

$$u'' + \lambda u(1-u^2) = 0 \qquad \text{(CI)}$$
$$u'(0) = u'(\pi) = 0.$$

For this (Chaffee-Infante) equation there exist bifurcating branches of solutions at $\lambda = \mu_n = n^2,\ n = 1, 2, \cdots,$ the n^{th} branch consisting of functions with n zeros, existing for all $\lambda > n^2$, and becoming asymptotic to "± 1".



Define $G(\lambda, u) : \mathbb{R} \times L^2 \to L^2$ by

$$G(\lambda, u) = L_\varepsilon u + \lambda f(u),$$

where $f(u) = u - u^3$, suitably cut off for $|u| > 1$. Note that $G(\lambda, 0) \equiv 0$, for all $\lambda \in \mathbb{R}$.

Lemma 3 *Assume $\lambda \in \sigma(L_\epsilon)$ and $dim(N(G_u(\lambda, 0))) < \infty$, then $G_u(\lambda, 0)$ is a Fredholm operator with index zero.*

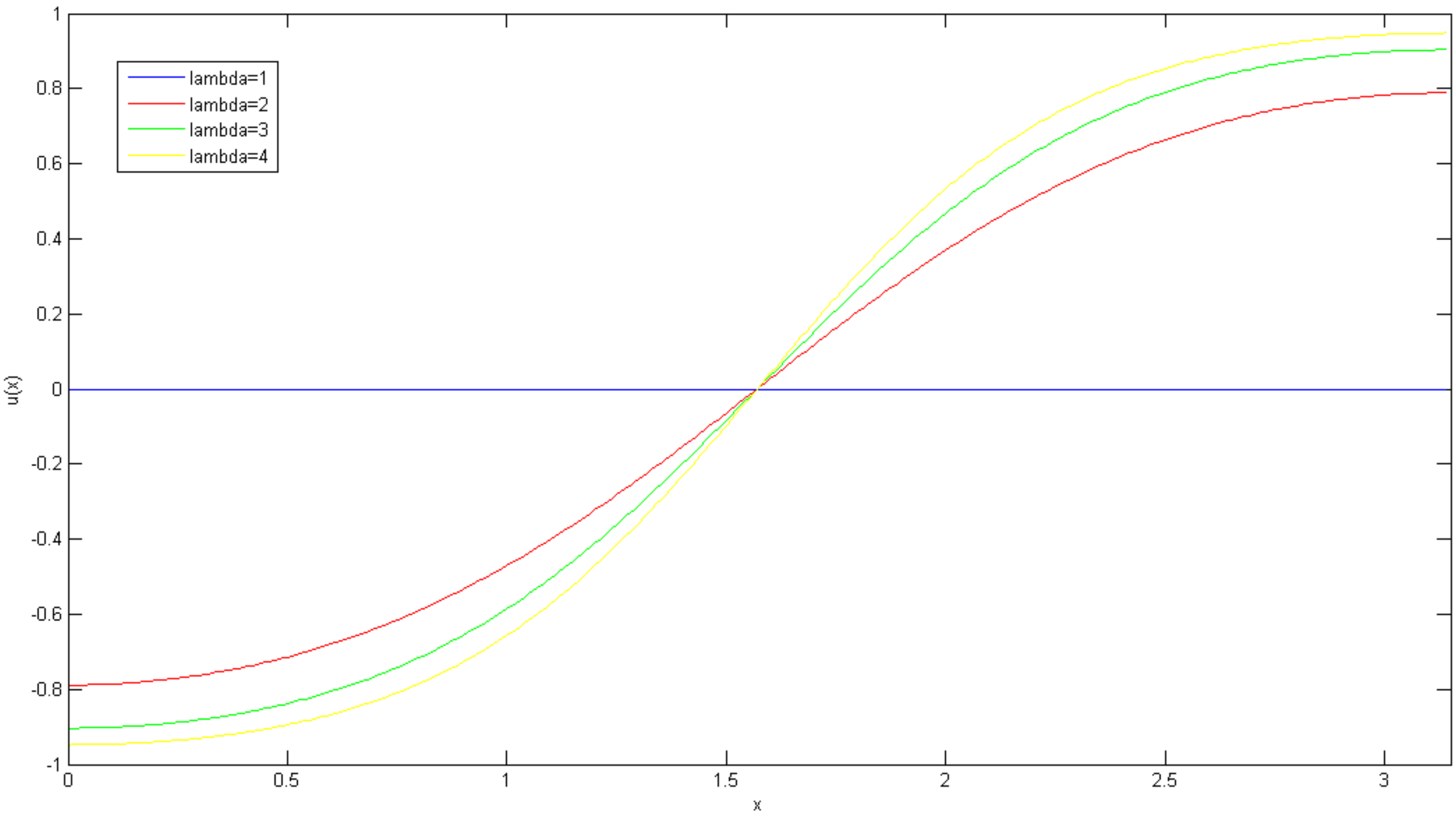
Thanks to a bifurcation theorem by Junping Shi and Xuefeng Wang and the results above

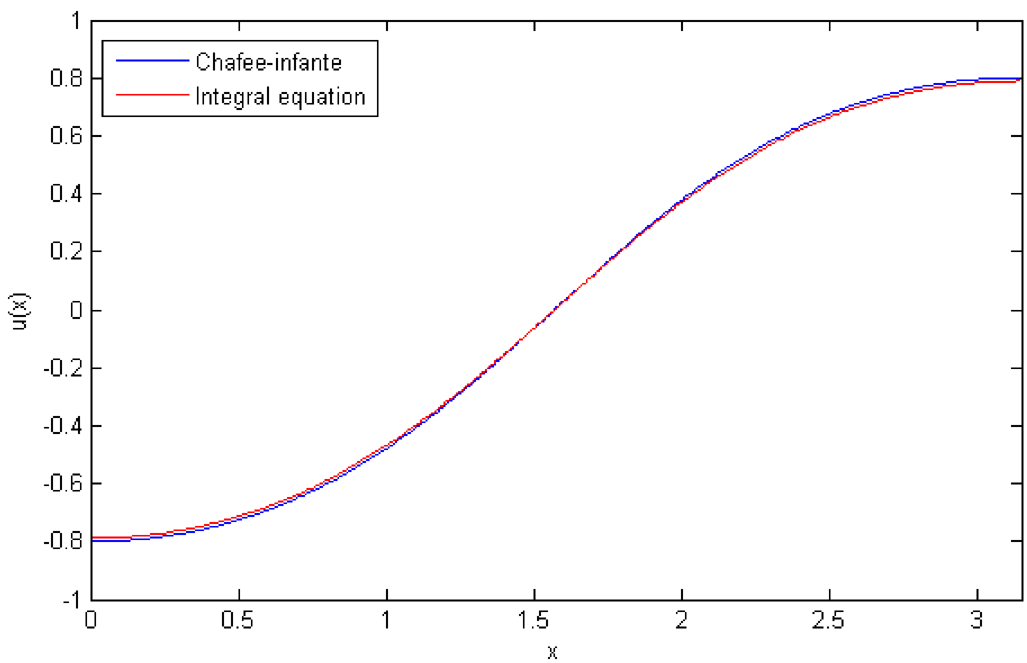
Theorem 2 *Suppose that $\mu \in \sigma(c_J \triangle^N)$ is a simple eigenvalue of $c_J \triangle^N$ (always true in 1-D). And let*
$B_\delta(\mu) = \{\lambda \in \mathbb{C} : |\lambda - \mu| < \delta\}$ with $\delta > 0$ so small that $B_\delta \cap \sigma(c_J \triangle^N) = \{\mu\}$. Then
(a) there exists $\epsilon_{\mu,\delta} > 0$ so that when $\epsilon \leq \epsilon_{\mu,\delta}$ $B_\delta \cap \sigma(L_\epsilon) = \{\lambda_0\}$ a simple eigenvalue of L_ϵ.
(b) if we write $\mathcal{N}(G_u(\lambda, 0)) = span\{\omega_0\}$ and Z is any complement of $\mathcal{N}(G_u(\lambda, 0))$ in $L^2(\Omega)$, then the solution set of $G(\lambda, u) = 0$ near $(\lambda_0, 0)$ consists precisely of the curves $u = 0$ and $\{(\lambda(s), u(s)) : s \in I = (-a, a)\}$, where $\lambda : I \mapsto \mathbb{R}$ is a C^2 function and $z : I \mapsto Z$ is a C^1 function such that $u(s) = s\omega_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$ and $\lambda'(0) = 0$.

Let μ_n, $n = 1, 2, \ldots$ be the positive eigenvalues of

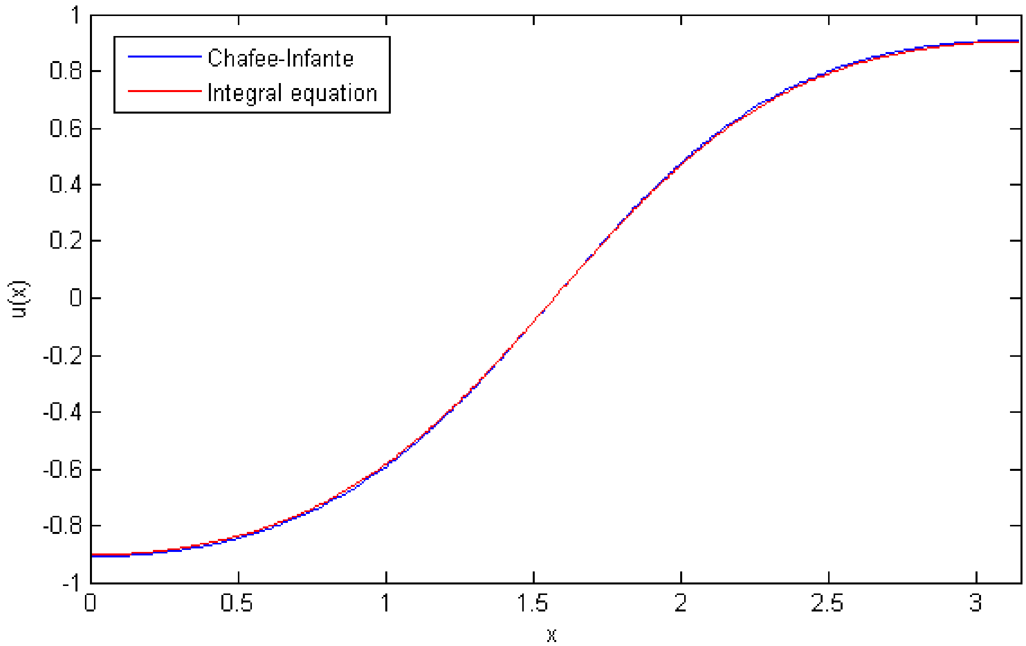
$$-c_J \phi'' = \mu \phi \ \text{ on } \ [0, \pi], \quad \phi'(0) = \phi'(\pi) = 0.$$

Corollary 1 *For fixed N and $\delta > 0$, there exists $\varepsilon_{N,\delta} > 0$ such that for all $0 < \varepsilon \leq \varepsilon_{N,\delta}$, in each $B_\delta(\mu_n)$, $n = 1, 2, \ldots, N$, there exists a simple eigenvalue, λ_n^ε, of $-L_\varepsilon$ with corresponding eigenfunction ϕ_n^ε and nontrivial solution branches to (NIC) of the form $\{(\lambda_n^\varepsilon(s), u_n^\varepsilon(s)) : s \in I_n^\varepsilon = (-a_n^\varepsilon, a_n^\varepsilon)\}$, where $\lambda_n^\varepsilon : I_n^\varepsilon \mapsto \mathbb{R}$ is a C^2 function and $z_n^\varepsilon : I \mapsto Z_n^\varepsilon$ is a C^1 function such that $u_n^\varepsilon(s) = s\phi_n^\varepsilon + s z_n^\varepsilon(s)$, $\lambda_n^\varepsilon(0) = \lambda_n^\varepsilon$, $z_n^\varepsilon(0) = 0$ and ${\lambda_n^\varepsilon}'(0) = 0$.*

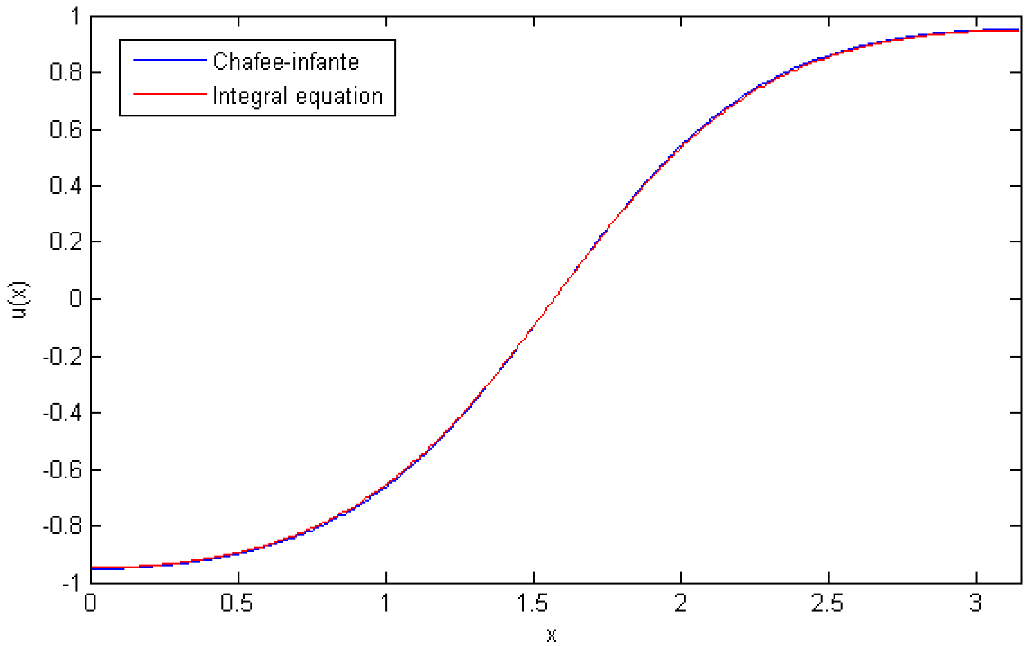




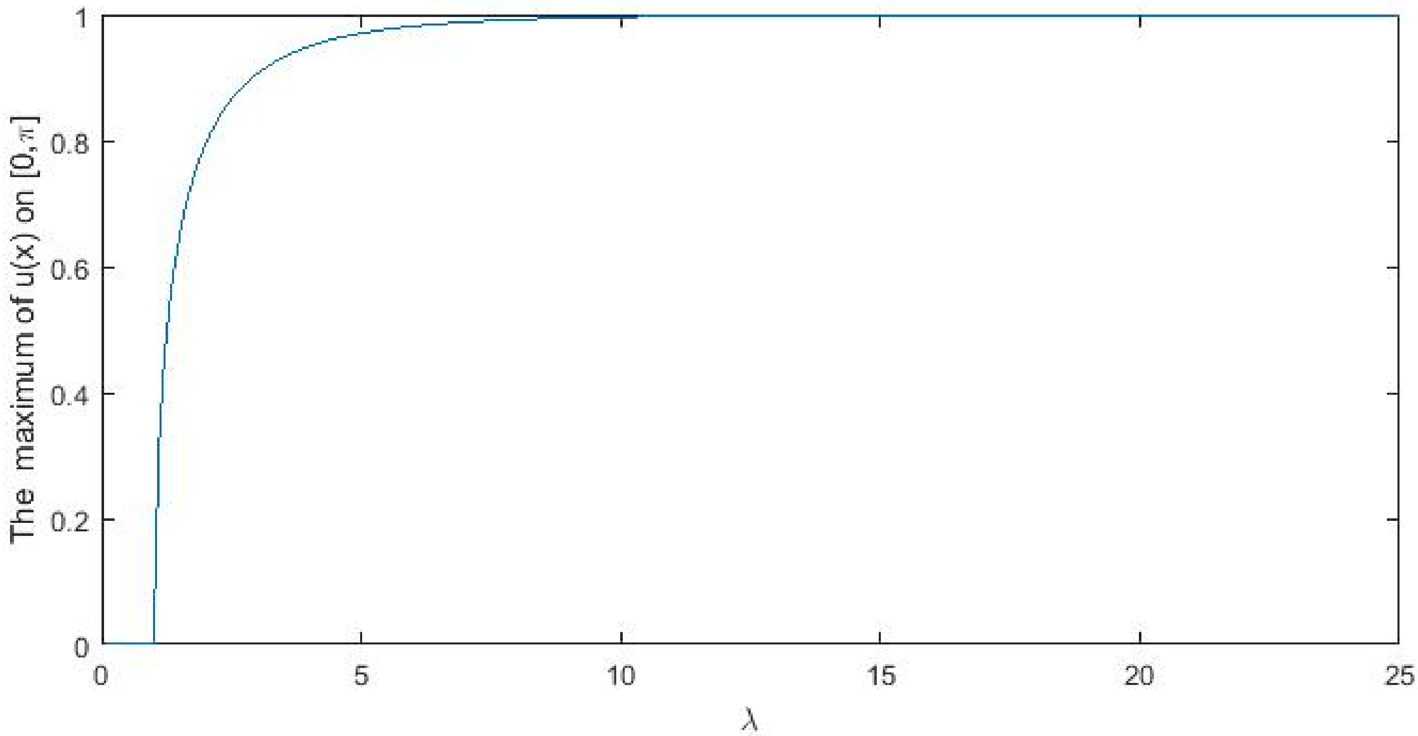
$\lambda = 2, \varepsilon = .02$

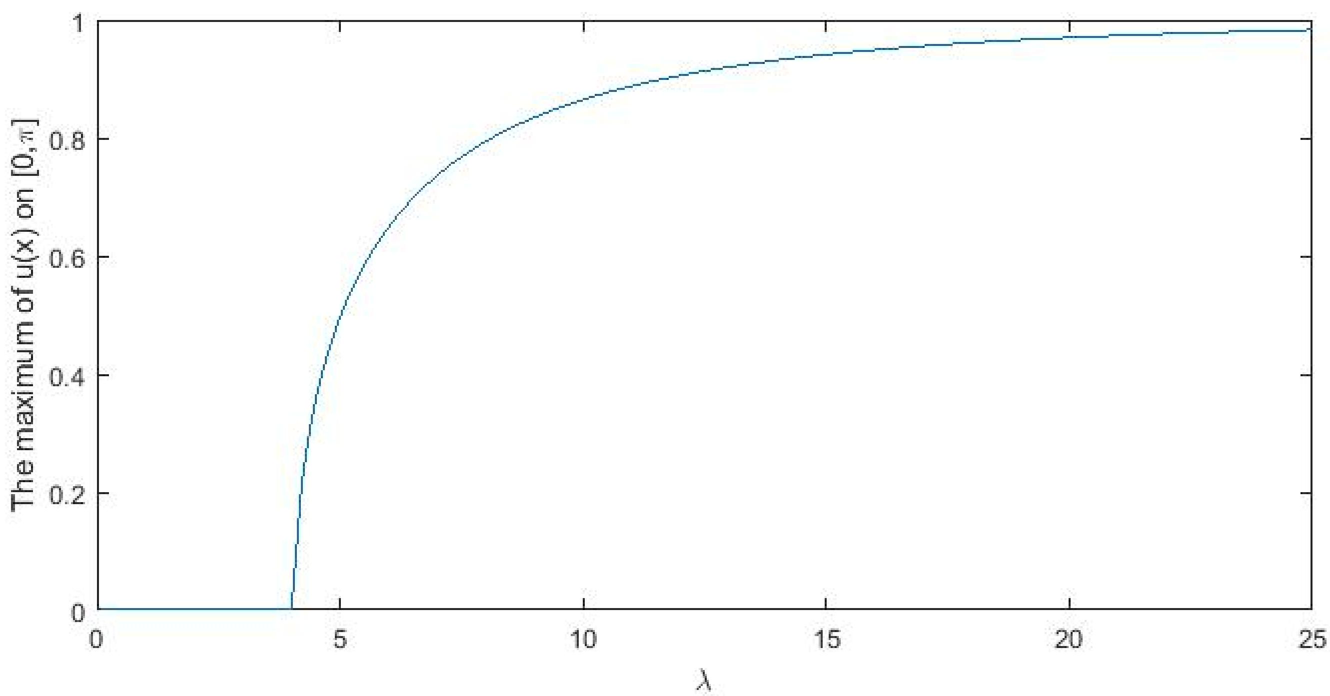


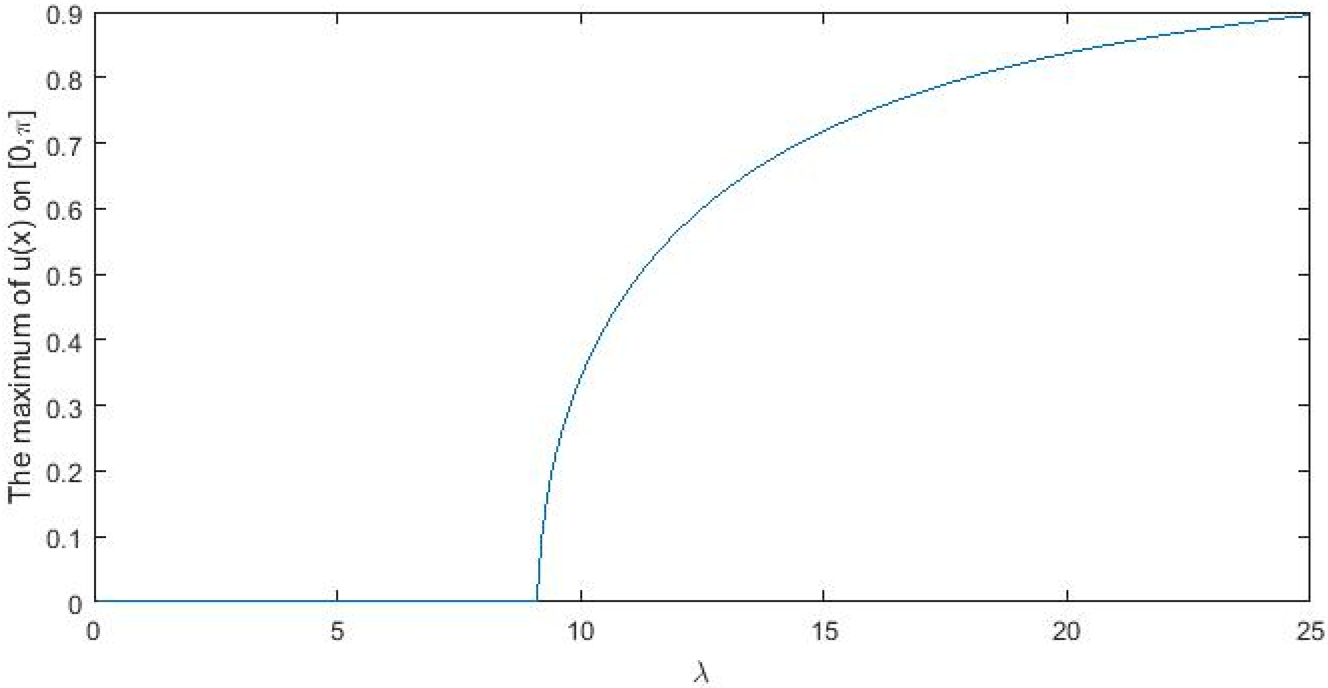
$\lambda = 3$



$\lambda = 4$


The maximum of u(x) on [0,π]
λ
0
5
10
15
20
25
0
0.2
0.4
0.6
0.8
1

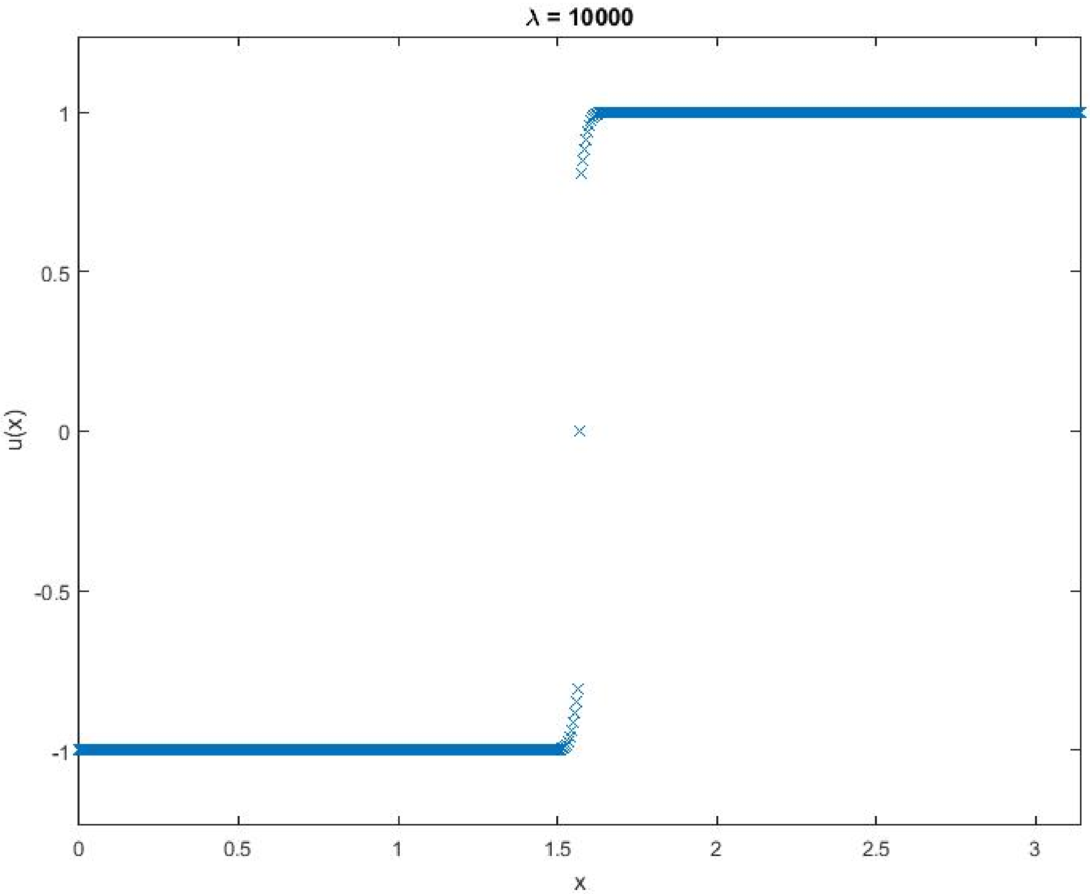

The maximum of u(x) on [0,π]
λ
0
5
10
15
20
25
0
0.2
0.4
0.6
0.8
1



What about global bifurcating branches?

Numerically–Yes. Proof missing.

Note that for large λ, solutions lying on the bifurcating branches are *discontinuous*.



To see why this must be so, recall

$$L_\varepsilon u + \lambda u(1-u^2) = 0 \quad \text{on } [0,\pi], \qquad \text{(NCI)}$$

Fix $\varepsilon > 0$. Now make $\varepsilon^2\lambda$ huge, > 10, e.g., by taking λ HUGER and try finding where $u = .5$, say.
The issue is, unlike the Laplacian, $\varepsilon^2 L_\varepsilon$ is uniformly bounded!

Open problem 2. Prove global bifurcation.

Thank you   Muchas Gracias